Homework 3 Solutions

February 9, 2022

Problem 4

Let R be the rectangle $R = \{(x_1, \dots, x_n) : a_i \leq x_i \leq b_i\}$, and let $\mathbf{z}$ and $\mathbf{w}$ be points in R. Then in particular for each $i = 1, \dots, n$ we have $a_i \leq z_i \leq b_i$ and $a_i \leq w_i \leq b_i$. Now let $\mathbf{y} = \lambda\mathbf{z} + (1-\lambda)\mathbf{w}$ with $\lambda \in [0,1]$ be any point on the line segment from $\mathbf{z}$ to $\mathbf{w}$. The *ith* coordinate of $\mathbf{y}$ is $y_i = \lambda w_i + (1-\lambda)z_i$. We notice that since λ and $1-\lambda$ are nonnegative we have

$$\begin{aligned} \lambda w_i + (1-\lambda)z_i &\leq \lambda b_i + (1-\lambda)b_i \\ &= b_i \end{aligned}$$

and

$$\begin{aligned} \lambda w_i + (1-\lambda)z_i &\geq \lambda a_i + (1-\lambda)a_i \\ &= a_i \end{aligned}$$

So we see that for each $i = 1, \dots, n$ we have $a_i \leq y_i \leq b_i$. We conclude that $\mathbf{y}$ is in R. As $\mathbf{y}$ was any point on the line segment between $\mathbf{z}$ and $\mathbf{w}$, which were themselves any two points in R, we conclude that R is convex.

An alternate solution is to observe that a rectangle is the set of the convex combinations of its extreme points (the 2^n corners) and apply the result from class that the set of convex combinations of finitely-many points in $\mathbb{R}^n$ is itself a convex set.

Problem 5

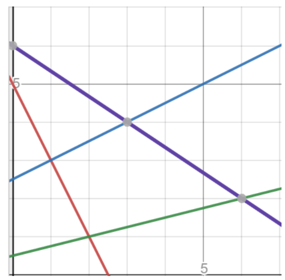
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. Then suppose that $\mathbf{z}$ and $\mathbf{w}$ are elements in the image $f(S) \subseteq \mathbb{R}^m$. The points on the line segment connecting $\mathbf{z}$ to $\mathbf{w}$ are $\lambda\mathbf{z} + (1-\lambda)\mathbf{w}$ for $\lambda \in [0,1]$. Now, we notice that since $\mathbf{z}$ and $\mathbf{w}$ are in the image of S, there are some $\mathbf{x}$ and $\mathbf{y}$ in S such that $f(\mathbf{x}) = \mathbf{z}$ and $f(\mathbf{y}) = \mathbf{w}$. Moreover, since S is convex, all of the points of the form $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ for $\lambda \in [0,1]$ lie in S. But if we apply f to a point of that form we get

$$\begin{aligned} f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}) \\ &= \lambda\mathbf{z} + (1-\lambda)\mathbf{w}. \end{aligned}$$

We therefore conclude that every point on the line segment between $\mathbf{z}$ and $\mathbf{w}$ lies in $f(S)$. Since $\mathbf{z}$ and $\mathbf{w}$ were any two points in $f(S)$, we conclude that $f(S)$ is convex.

Problem 6

The sketch is below; the set of convex combinations is the four sided region in the center. The extreme points are $(1,3)$, $(2,1)$, $(3,4)$, and $(6,2)$.



We may write

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

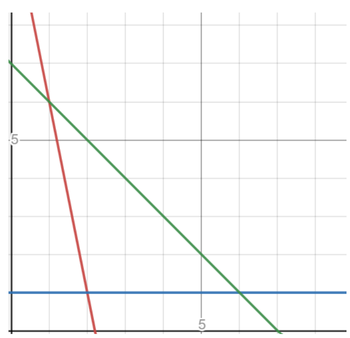
or alternately

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{4}{9}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{4}{9}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

There are infinitely many additional correct solutions. (If you're bored, confirm that fact using what you know about linear algebra.)

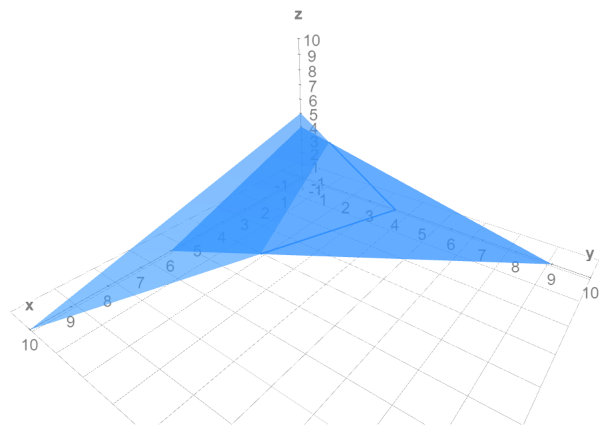
Problem 7

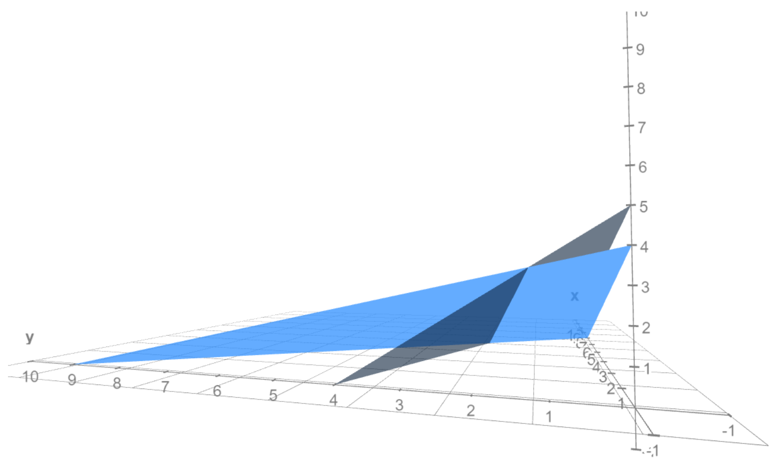
The convex combinations of the four points form the triangle in the sketch shown below. The extreme points are $(1,6)$, $(2,1)$, and $(6,1)$.



Problem 8

The sketch is below; we are looking at the five-sided region under both planes. A view from the back looking directly onto the region is included for clarity.





We see that there are six extreme points. Four of them are easily located: $(0,0,0)$, $(6,0,0)$, $(0,4,0)$, and $(0,0,4)$. For the remaining two we must do some algebra. We see that one of the extreme points occurs at the intersections of the planes $6x_1+4x_2+9x_3=36$, $2x_1+5x_2+4x_3=20$, and $x_1=0$. So we need to solve

$$\begin{cases} 4x_2+9x_3=36 \\ 5x_2+4x_3=20 \end{cases}$$

The intersection of these lines is at $x_2=\frac{36}{29}$ and $x_3=\frac{100}{29}$, so the extreme point is $\left(0,\frac{36}{29},\frac{100}{29}\right)$. The final extreme point occurs at the intersection of $6x_1+4x_2+9x_3=36$, $2x_1+5x_2+4x_3=20$, and $x_3=0$. So we need to solve

$$\begin{cases} 6x_1+4x_2=36 \\ 2x_1+5x_2=20 \end{cases}$$

The intersection of these lines is at $x_1=\frac{50}{11}$ and $x_2=\frac{24}{11}$, so the final extreme point is at $\left(\frac{50}{11},\frac{24}{11},0\right)$. We sanity-check our algebra by looking at the graph and noting that these numbers seem to be about right.

Citation: Graphs from Desmos and Math3d.