Problem 4

Let $R$ be the rectangle $R = \{(x_1, \ldots, x_n) : a_i \leq x_i \leq b_i\}$, and let $z$ and $w$ be points in $R$. Then in particular for each $i = 1, \ldots, n$ we have $a_i \leq z_i \leq b_i$ and $a_i \leq w_i \leq b_i$. Now let $y = \lambda z + (1 - \lambda)w$ with $\lambda \in [0, 1]$ be any point on the line segment from $z$ to $w$. The $i$th coordinate of $y$ is $y_i = \lambda w_i + (1 - \lambda)z_i$. We notice that since $\lambda$ and $1 - \lambda$ are nonnegative we have

$$\lambda w_i + (1 - \lambda)z_i \leq \lambda b_i + (1 - \lambda)b_i$$

$$= b_i$$

and

$$\lambda w_i + (1 - \lambda)z_i \geq \lambda a_i + (1 - \lambda)a_i$$

$$= a_i$$

So we see that for each $i = 1, \ldots, n$ we have $a_i \leq y_i \leq b_i$. We conclude that $y$ is in $R$. As $y$ was any point on the line segment between $z$ and $w$, which were themselves any two points in $R$, we conclude that $R$ is convex.

An alternate solution is to observe that a rectangle is the set of the convex combinations of its extreme points (the $2^n$ corners) and apply the result from class that the set of convex combinations of finitely-many points in $\mathbb{R}^n$ is itself a convex set.

Problem 5

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. Then suppose that $z$ and $w$ are elements in the image $f(S) \subseteq \mathbb{R}^m$. The points on the line segment connecting $z$ to $w$ are $\lambda z + (1 - \lambda)w$ for $\lambda \in [0, 1]$. Now, we notice that since $z$ and $w$ are in the image of $S$, there are some $x$ and $y$ in $S$ such that $f(x) = z$ and $f(y) = w$. Moreover, since $S$ is convex, all of the points of the form $\lambda x + (1 - \lambda)y$ for $\lambda \in [0, 1]$ lie in $S$. But if we apply $f$ to a point of that form we get

$$f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y)$$

$$= \lambda z + (1 - \lambda)w.$$ 

We therefore conclude that every point on the line segment between $z$ and $w$ lies in $f(S)$. Since $z$ and $w$ were any two points in $f(S)$, we conclude that $f(S)$ is convex.
Problem 6

The sketch is below; the set of convex combinations is the four sided region in the center. The extreme points are (1, 3), (2, 1), (3, 4), and (6, 2).

We may write

\[
\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 6 \\ 2 \end{bmatrix}
\]

or alternately

\[
\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{4}{9} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{4}{9} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 6 \\ 2 \end{bmatrix}
\]

There are infinitely many additional correct solutions. (If you’re bored, confirm that fact using what you know about linear algebra.)

Problem 7

The convex combinations of the four points form the triangle in the sketch shown below. The extreme points are (1, 6), (2, 1), and (6, 1).

Problem 8

The sketch is below; we are looking at the five-sided region under both planes. A view from the back looking directly onto the region is included for clarity.
We see that there are six extreme points. Four of them are easily located: (0,0,0), (6,0,0),
(0,4,0), and (0,0,4). For the remaining two we must do some algebra. We see that one of the
extreme points occurs at the intersections of the planes 6x_1+4x_2+9x_3 = 36, 2x_1+5x_2+4x_3 = 20,
and x_1 = 0. So we need to solve
\[
\begin{align*}
4x_2 + 9x_3 &= 36 \\
5x_2 + 4x_3 &= 20
\end{align*}
\]
The intersection of these lines is at $x_2 = \frac{36}{29}$ and $x_3 = \frac{100}{29}$, so the extreme point is \((0, \frac{36}{29}, \frac{100}{29})\).
The final extreme point occurs at the intersection of $6x_1+4x_2+9x_3 = 36$, $2x_1+5x_2+4x_3 = 20$, and $x_3 = 0$. So we need to solve
\[
\begin{align*}
6x_1 + 4x_2 &= 36 \\
2x_1 + 5x_2 &= 20
\end{align*}
\]
The intersection of these lines is at $x_1 = \frac{50}{11}$ and $x_2 = \frac{24}{11}$, so the final extreme point is at \((\frac{50}{11}, \frac{24}{11}, 0)\). We sanity-check our algebra by looking at the graph and noting that these numbers seem to be about right.

Citation: Graphs from Desmos and Math3d.