Homework 2 Solutions

February 3, 2022

Problem 4

The problem is to maximize

$$z = 20x_1 + 30x_2$$

subject to the constraints

$$\begin{cases} 2x_1 + x_2 \leq 480 \\ 2x_1 + 3x_2 \leq 640 \\ x_1, x_2 \geq 0 \end{cases}.$$

This is already in standard form. Equivalently, in matrix notation, we may write that we wish to maximize

$$z = \begin{bmatrix} 20 & 30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to the constraints

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 480 \\ 640 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In order to put the problem in canonical form, we may add two slack variables u_1 and u_2 such that the problem is to maximize

$$z = 20x_1 + 30x_2$$

subject to the constraints

$$\begin{cases} 2x_1 + x_2 + u_1 = 480 \\ 2x_1 + 3x_2 + u_2 = 640 \\ x_1, x_2, u_1, u_2 \geq 0 \end{cases}.$$

Equivalently we may write in matrix notation that we wish to maximize

$$z = \begin{bmatrix} 20 & 30 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{bmatrix}$$

subject to the constraints

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 480 \\ 640 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The optimal solution from the last homework is any point (x_1, x_2) along the line $2x_1+3x_2 = 640$ between $\left(0, \frac{640}{3}\right)$ and $(200, 80)$. We observe that everywhere along this line the slack variable u_2 is 0, and $u_1 = 2x_2 - 160$, which ranges from $u_1 = \frac{800}{3}$ at the point $\left(0, \frac{640}{3}\right)$ to $u_1 = 0$ at the intersection point $(20, 80)$. This means that in every optimal solution we use all of Coffee C, and in any given optimal solution the number u_1 is four times the number of kilograms of Coffee B left over (because we multiplied by four in the course of making the constraints more tractable).

Problem 5

The problem is to minimize

$$z = 15,000x_1 + 20,000x_2$$

subject to the constraints

$$\begin{cases} x_1 + 2x_2 \leq 12 \\ 40x_1 + 50x_2 \geq 360 \\ x_1, x_2 \geq 0 \end{cases}.$$

First we need to change this to a maximization problem; ergo, we will look for the maximum of $z = -15,000x - 20,000x_2$. We also multiply the second constraint by -1, obtaining the equivalent constraint $-40x_1 - 50x_2 \leq -360$. Then in standard form the problem is to maximize

$$z = -15,000x_1 - 20,000x_2$$

subject to the constraints

$$\begin{cases} x_1 + 2x_2 \leq 12 \\ -40x_1 - 50x_2 \leq -360 \\ x_1, x_2 \geq 0 \end{cases}.$$

Equivalently, in matrix notation we may write that we wish to maximize

$$z' = \begin{bmatrix} -15,000 & -20,000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to the constraints

$$\begin{bmatrix} 1 & 2 \\ -40 & -50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 12 \\ -360 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In order to put this problem into canonical form we add two slack variables u_1 and u_2, making the problem to maximize

$$z = -15,000x_1 - 20,000x_2$$

subject to the constraints

$$\begin{cases} x_1 + 2x_2 + u_1 = 12 \\ -40x_1 - 50x_2 + u_2 = -360 \\ x_1, x_2, u_1, u_2 \geq 0 \end{cases}.$$

which we may equivalent write in matrix notation as wishing to maximize

$$z = \begin{bmatrix} -15,000 & -20,000 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{bmatrix}$$

subject to the constraints

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -40 & -50 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -360 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From last week, the optimal solution occurs at $(x_1, x_2) = (9, 0)$. There we have $u_1 = 3$ and $u_2 = 0$, corresponding to not needing 3 workers and making exactly the right number of boxes.

Problem 6

The problem is to maximize $z = 2x_1 + 3x_2 + 5x_3$ subject to the constraints

$$\begin{cases} 4x_1 - x_2 \geq 3 \\ x_1 + x_2 + x_3 \leq 5x_1 \\ x_3 = 7 \\ x_1 \geq 0, x_3 \geq 0 \end{cases}.$$

We start by putting this in standard form. To do this, we start by replacing $4x_1 - x_2 \geq 3$ with $-4x_1 + x_2 \leq -3$. We also change $x_1 + x_2 + x_3 \leq 5x_1$ to $-4x_1 + x_2 + x_3 \leq 0$. We further replace $x_3 = 7$ by $x_3 \leq 7$ and $-x_3 \leq -7$. Finally, we observe that x_2 is unconstrained, so we replace $x_2 = x_2^+ - x_2^-$ with $x_2^+, x_2^- \geq 0$.

With all this in mind, in standard form our problem is to maximize $z = 2x_1 + 3x_2^+ - 3x_2^- + 5x_3$ subject to the constraints

$$\begin{cases} -4x_1 + x_2^+ - x_2^- \leq -3 \\ -4x_1 + x_2^+ - x_2^- + x_3 \leq 0 \\ x_3 \leq 7 \\ -x_3 \leq -7 \\ x_1, x_2^+, x_2^-, x_3 \geq 0 \end{cases}.$$

Equivalently, in matrix notation we may write that we wish to maximize

$$z = \begin{bmatrix} 2 & 3 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2^+ \\ x_2^- \\ x_3 \end{bmatrix}$$

subject to the constraints

$$\begin{bmatrix} -4 & 1 & -1 & 0 \\ -4 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2^+ \\ x_2^- \\ x_3 \end{bmatrix} \leq \begin{bmatrix} -3 \\ 0 \\ 7 \\ -7 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2^+ \\ x_2^- \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now we put the problem into canonical form. We may as well change back to having $x_3 = 7$ instead of the last two inequalities. We also introduce two slack variables u_1 and u_2 to make the other two inequalities into equalities, after which the problem is to maximize $z = 2x_1 + 3x_2^+ - 3x_2^- + 5x_3$ subject to the constraints

$$\begin{cases} -4x_1 + x_2^+ - x_2^- + u_1 = -3 \\ -4x_1 + x_2^+ - x_2^- + x_3 + u_2 = 0 \\ x_3 = 7 \\ x_1, x_2^+, x_2^-, x_3, u_1, u_2 \geq 0 \end{cases}.$$

Equivalently in matrix notation we may write that we wish to maximize

$$z = \begin{bmatrix} 2 & 3 & -3 & 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2^+ \\ x_2^- \\ x_3 \\ u_1 \\ u_2 \end{bmatrix}$$

subject to the constraints

$$\begin{bmatrix} -4 & 1 & -1 & 0 & 1 & 0 \\ -4 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2^+ \\ x_2^- \\ x_3 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2^+ \\ x_2^- \\ x_3 \\ u_1 \\ u_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 7

We may consider for example attempting to maximize $z = 3x_1 + x_2$ subject to the constraints

$$\begin{cases} x_1 + x_2 \leq -3 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

which clearly has no feasible solutions since $x_1 + x_2$ is positive.

Problem 8

Given an $m \times n$ matrix $\mathbf{A}$ and a vector $\mathbf{x}$ of length n, it is easy to remind oneself that $\mathbf{A}(k\mathbf{x}) = k\mathbf{A}\mathbf{x}$ and $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y}$. So, in particular, we see that

$$\begin{aligned} \mathbf{A}(r\mathbf{x} + s\mathbf{y}) &= r\mathbf{A}\mathbf{x} + s\mathbf{A}\mathbf{y} \\ &= r\mathbf{b} + s\mathbf{b} \\ &= (r + s)\mathbf{b} \\ &= \mathbf{b} \end{aligned}$$

Moreover, recall that the entries of $\mathbf{x}$ and $\mathbf{y}$ are positive. In particular if x_i is the ith entry of $\mathbf{x}$ and y_i is the ith entry of $\mathbf{y}$, we have that $x_i, y_i \geq 0$, so we see that $rx_i + sy_i \geq 0$ since $r, s \geq 0$. So $r\mathbf{x} + s\mathbf{y} \geq \mathbf{0}$. We conclude that $r\mathbf{x} + s\mathbf{y}$ is a feasible solution to the linear programming problem. Note that this problem was accidentally described as being in standard form despite the equations being in canonical form; the proof works for either form with $=$ replaced by $\leq$ in the second line of the equation above.