Homework 1 Solutions

January 24, 2022

Problem 3

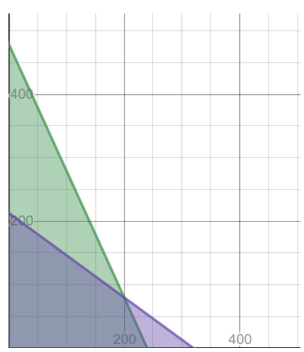
Let x_1 be the amount of Super blend that is made, x_2 be the amount of Deluxe brand that is made, and z be the total profit. We see that the objective function to be maximized is the profit, which we will write in cents

$$z = 20x_1 + 30x_2$$

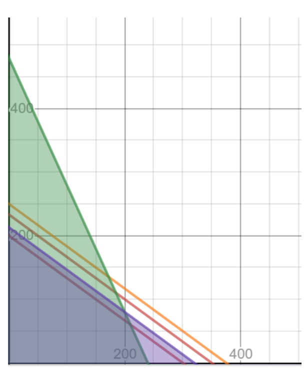
subject to the constraints

$$\begin{cases} .5x_1 + .25x_2 \leq 120 \\ .5x_1 + .75x_2 \leq 160 \\ x_1, x_2 \geq 0 \end{cases}$$

Indeed, we may multiply the first two constraints by 4 to obtain $2x_1 + x_2 \leq 480$ and $2x_1 + 3x_2 \leq 640$, which is somewhat easier to graph. The region of feasible solutions is the region shaded in both purple and green below.



We graph a few level sets of $z = 20x_1 + 30x_2$; the lines shown here are the level sets $z = 7500$, $z = 7000$, and $z = 6500$.



We see that we expect the maximum to be any point along the line $2x_1 + 3x_2 = 640$ between the points $(0, \frac{640}{3})$ and $(200, 80)$. The profit at any point along this line is \$64.

Problem 4

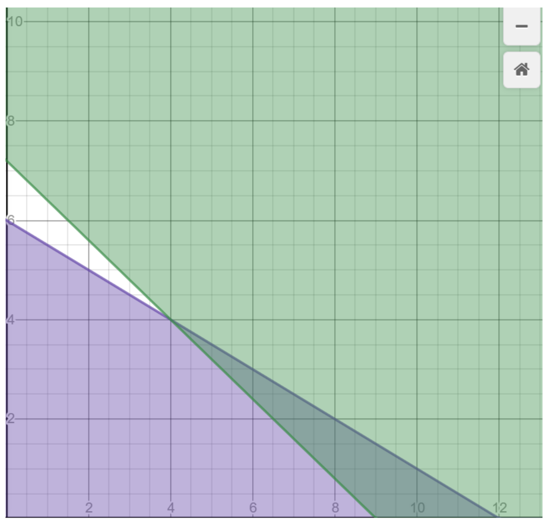
Let x_1 be the number of model A machines, x_2 be the number of model B machines, and the cost be z. Then the objective function to be minimized is

$$z = 15,000x_1 + 20,000x_2$$

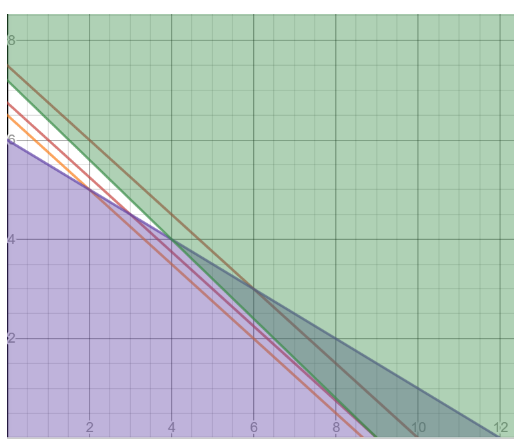
subject to the constraints

$$\begin{cases} x_1 + 2x_2 \leq 12 \\ 40x_1 + 50x_2 \geq 360 \\ x_1, x_2 \geq 0 \end{cases}$$

The set of feasible solutions is the region shaded in green and purple shown below



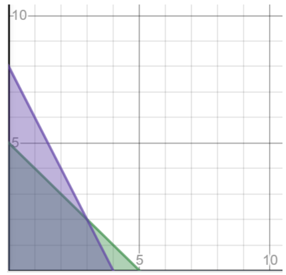
We now graph a few level sets of $z = 15,000x_1 + 20,000x_2$, specifically the lines $z = 150,000$, $z = 135,000$, and $z = 130,000$.



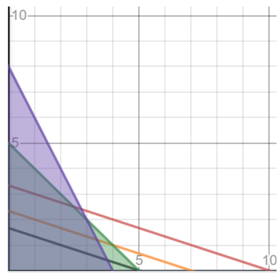
We see that the cost is minimized at the intersection of $x_2 = 0$ and $40x_1 + 50x_2 = 360$, which is $(9, 0)$. So the manufacturer should purchase 9 copies of machine A and pay \$135,000.

Problem 5

The set of feasible solutions is the region shaded in green and purple below



We graph a few level sets of $z_1 = x + 3y$, specifically the lines $z = 10$, $z = 7$, and $z = 5$ below.



We see that the function $z_1 = x + 3y$ is maximized at the corner $(0, 5)$, where $z_1 = 15$. Now we instead graph some level sets of $z_2 = 5x + 3y$, namely $z_2 = 30$, $z_2 = 25$, and $z_2 = 20$.

We see the function $z_2 = 5x + 3y$ is maximized at the corner given by the intersection point of the lines $x + y = 5$ and $2x + y = 8$, which is the point $(3, 2)$, where $z_2 = 15 + 6 = 21$.

Problem 6

We let x_1 be the number of paperback editions, x_2 be the number of book club editions, and x_3 be the number of library editions. Then the objective function to be maximized is the profit (given in dollars)

$$z = .5x_1 + .8x_2 + 1.2x_3$$

and the constraints are

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 \leq 420 \\ 4x_1 + 6x_2 + 10x_3 \leq 600 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Note that we had to change the availability of the gluing and sewing process to be in minutes for consistency.

Problem 7

We let x_1 be the amount of money invested in the utilities stock, x_2 be the amount of money invested in the electronics stock, and x_3 be the amount of money invested in the bond. Then the objective function to be maximized is the return z given by

$$z = .09x_1 + .04x_2 + .05x_3$$

and the constraints are

$$\begin{cases} x_1 + x_2 + x_3 \leq 200000 \\ x_1 + x_2 - x_3 \leq 0 \\ x_1 \leq 40000 \\ x_3 \geq 70000 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

where the second constraint comes from the fact that stocks must be less than half the money invested, that is, that $x_1 + x_2 \leq .5(x_1 + x_2 + x_3)$, which can be rearranged to the equation shown.

Citation: Images produced using the helpful graph plotter at `transum.org`.