Homework 13 Solutions

April 23, 2022

Problem 5

We begin with the following transportation tableau.

$\boxed{5}$	$\boxed{5}$	$\boxed{7}$	$\boxed{2}$	50
$\boxed{4}$	$\boxed{8}$	$\boxed{4}$	$\boxed{2}$	80
$\boxed{6}$	$\boxed{2}$	$\boxed{3}$	$\boxed{1}$	40
$\boxed{9}$	$\boxed{5}$	$\boxed{4}$	$\boxed{3}$	40
75	50	60	25	

We now attempt to use Vogel's method to fill in an initial solution. We compute the difference in costs between the cheapest and second cheapest route in each row and column and find that the largest is a tie between the first row and second column, both at 3. So we fill in the first row.

$\boxed{5}$ 25	$\boxed{5}$ 0	$\boxed{7}$ 0	$\boxed{2}$ 25	50
$\boxed{4}$	$\boxed{8}$	$\boxed{4}$	$\boxed{2}$ 0	80
$\boxed{6}$	$\boxed{2}$	$\boxed{3}$	$\boxed{1}$ 0	40
$\boxed{9}$	$\boxed{5}$	$\boxed{4}$	$\boxed{3}$ 0	40
75	50	60	25	

Ignoring the occupied entries, we see that among the rows and columns the largest difference between cheapest route and second cheapest route is now in the second column, at 3. So we fill in the second column.

$\boxed{5}$ 25	$\boxed{5}$ 0	$\boxed{7}$ 0	$\boxed{2}$ 25	50
$\boxed{4}$	$\boxed{8}$ 0	$\boxed{4}$	$\boxed{2}$ 0	80
$\boxed{6}$ 0	$\boxed{2}$ 40	$\boxed{3}$ 0	$\boxed{1}$ 0	40
$\boxed{9}$	$\boxed{5}$ 10	$\boxed{4}$	$\boxed{3}$ 0	40
75	50	60	25	

Again ignoring the occupied entries, we see that the largest difference between cheapest and second cheapest route in any row or column occurs in the first column or fourth row, which give the same resulting tableau, shown below.

$\boxed{5}$ 25	$\boxed{5}$ 0	$\boxed{7}$ 0	$\boxed{2}$ 25	50
$\boxed{4}$ 50	$\boxed{8}$ 0	$\boxed{4}$ 30	$\boxed{2}$ 0	80
$\boxed{6}$ 0	$\boxed{2}$ 40	$\boxed{3}$ 0	$\boxed{1}$ 0	40
$\boxed{9}$ 0	$\boxed{5}$ 10	$\boxed{4}$ 30	$\boxed{3}$ 0	40
75	50	60	25	

There are seven nonzero values, so this solution is nondegenerate. We now attempt to solve the corresponding dual problem to determine the entering variable. We have the following equations for the basic variables.

$$\begin{cases} u_1 + v_1 = 5 \\ u_1 + v_4 = 2 \\ u_2 + v_1 = 4 \\ u_2 + v_3 = 4 \\ u_3 + v_2 = 2 \\ u_4 + v_2 = 5 \\ u_4 + v_3 = 4 \end{cases}$$

If we let $u_1 = 0$, these equations are solved by

$$u_1 = 0 \qquad u_2 = -1 \qquad u_3 = -4 \qquad u_4 = -1$$

$$v_1 = 5 \qquad v_2 = 6 \qquad v_3 = 5 \qquad v_4 = 2.$$

We get the following values for the notional objective row entry (called the reduced cost) associated to each nonbasic variable.

$$\begin{aligned}
x_{12} &: u_1 + v_2 - c_{12} = 0 + 6 - 5 = 1 \\
x_{13} &: u_1 + v_3 - c_{13} = 0 + 5 - 7 = -2 \\
x_{22} &: u_2 + v_2 - c_{22} = -1 + 6 - 8 = -3 \\
x_{24} &: u_2 + v_4 - c_{24} = -1 + 2 - 2 = -1 \\
x_{31} &: u_3 + v_1 - c_{31} = -4 + 5 - 6 = -5 \\
x_{33} &: u_3 + v_3 - c_{33} = -4 + 5 - 3 = -2 \\
x_{34} &: u_3 + v_4 - c_{34} = -4 + 2 - 1 = -3 \\
x_{41} &: u_4 + v_1 - c_{41} = -1 + 5 - 9 = -5 \\
x_{44} &: u_4 + v_4 - c_{44} = -1 + 2 - 3 = -2
\end{aligned}$$

So, the entering variable is x_{12}. This gives us the loop shown in the table, which makes the departing variable x_{42}.

$\boxed{5}$ - 25	$\boxed{5}$ + 0	$\boxed{7}$ 0	$\boxed{2}$ 25	50
$\boxed{4}$ + 50	$\boxed{8}$ 0	$\boxed{4}$ - 30	$\boxed{2}$ 0	80
$\boxed{6}$ 0	$\boxed{2}$ 40	$\boxed{3}$ 0	$\boxed{1}$ 0	40
$\boxed{9}$ 0	$\boxed{5}$ - 10	$\boxed{4}$ + 30	$\boxed{3}$ 0	40
75	50	60	25	

Pivoting accordingly we obtain the following.

$\boxed{5}$ 15	$\boxed{5}$ 10	$\boxed{7}$ 0	$\boxed{2}$ 25	50
$\boxed{4}$ 60	$\boxed{8}$ 0	$\boxed{4}$ 20	$\boxed{2}$ 0	80
$\boxed{6}$ 0	$\boxed{2}$ 40	$\boxed{3}$ 0	$\boxed{1}$ 0	40
$\boxed{9}$ 0	$\boxed{5}$ 0	$\boxed{4}$ 40	$\boxed{3}$ 0	40
75	50	60	25	

We now consider optimality of this tableau by studying the dual problem. The basic variables give us the following equations.

$$\begin{cases} u_1 + v_1 = 5 \\ u_1 + v_2 = 5 \\ u_1 + v_4 = 2 \\ u_2 + v_1 = 4 \\ u_2 + v_3 = 4 \\ u_3 + v_2 = 2 \\ u_4 + v_3 = 4 \end{cases}$$

Letting $u_1 = 0$ these equations are solved by

$$u_1 = 0 \qquad u_2 = -1 \qquad u_3 = -3 \qquad u_4 = -1$$

$$v_1 = 5 \qquad v_2 = 5 \qquad v_3 = 5 \qquad v_4 = 2.$$

The reduced costs associated to the nonbasic variables are therefore

$$\begin{aligned}
x_{13} &: u_1 + v_3 - c_{13} = 0 + 5 - 7 = -2 \\
x_{22} &: u_2 + v_2 - c_{22} = -1 + 5 - 8 = -4 \\
x_{24} &: u_2 + v_4 - c_{24} = -1 + 2 - 2 = -1 \\
x_{31} &: u_3 + v_1 - c_{31} = -3 + 5 - 6 = -4 \\
x_{33} &: u_3 + v_3 - c_{33} = -3 + 5 - 3 = -1 \\
x_{34} &: u_3 + v_4 - c_{34} = -3 + 2 - 1 = -2 \\
x_{41} &: u_4 + v_1 - c_{41} = -1 + 5 - 9 = -5 \\
x_{42} &: u_4 + v_2 - c_{42} = -1 + 5 - 5 = -1 \\
x_{44} &: u_4 + v_4 - c_{44} = -1 + 2 - 3 = -2
\end{aligned}$$

As these are all negative, we conclude the tableau is optimal. The minimum cost is $C = 5(15) + 5(10) + 2(25) + 4(60) + 4(20) + 2(40) + 4(40) = 735$.

Problem 6

We can model this as a balanced problem if we add a dummy fourth source, say s_4, that "ships" a total of 200 cars such that the cost of shipping a car from s_4 to Denver is \$200 and the cost of shipping a car from s_4 to Miami is \$300. This gives

$$C = \begin{bmatrix} 80 & 215 \\ 100 & 108 \\ 102 & 68 \\ 200 & 300 \end{bmatrix}, \qquad \mathbf{s} = \begin{bmatrix} 1000 \\ 1300 \\ 1200 \\ 200 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} 2300 \\ 1400 \end{bmatrix}$$

for a starting tableau of the following form.

$\boxed{80}$	$\boxed{215}$	1000
$\boxed{100}$	$\boxed{108}$	1300
$\boxed{102}$	$\boxed{68}$	1200
$\boxed{200}$	$\boxed{300}$	200
2300	1400	

Following Vogel's method we see that the row or column with largest difference between the cheapest and second cheapest route is the first row. So we fill it in.

$\boxed{80}$ 1000	$\boxed{215}$ 0	1000
$\boxed{100}$	$\boxed{108}$	1300
$\boxed{102}$	$\boxed{68}$	1200
$\boxed{200}$	$\boxed{300}$	200
2300	1400	

Ignoring the occupied entries, we see the row or column with greatest difference between the cheapest and second cheapest route is now the fourth row. We fill it in.

$\boxed{80}$ 1000	$\boxed{215}$ 0	1000
$\boxed{100}$	$\boxed{108}$	1300
$\boxed{102}$	$\boxed{68}$	1200
$\boxed{200}$ 200	$\boxed{300}$ 0	200
2300	1400	

The next worst difference is the second row, which we now fill in; this forces the remaining entries.

$\boxed{80}$ 1000	$\boxed{215}$ 0	1000
$\boxed{100}$ 1100	$\boxed{108}$ 200	1300
$\boxed{102}$ 0	$\boxed{68}$ 1200	1200
$\boxed{200}$ 200	$\boxed{300}$ 0	200
2300	1400	

We see five nonzero entries, so this solution is nondegenerate. Now we look at the dual problem to determine whether we could improve on this solution. The basic variables give us the equations

$$\begin{cases} u_1 + v_1 = 80 \\ u_2 + v_1 = 100 \\ u_2 + v_2 = 108 \\ u_3 + v_2 = 68 \\ u_4 + v_1 = 200 \end{cases}$$

Setting $u_1 = 0$ we see that these equations are solved by

$$u_1 = 0 \qquad u_2 = 20 \qquad u_3 = -20 \qquad u_4 = 120$$

$$v_1 = 80 \qquad v_2 = 88.$$

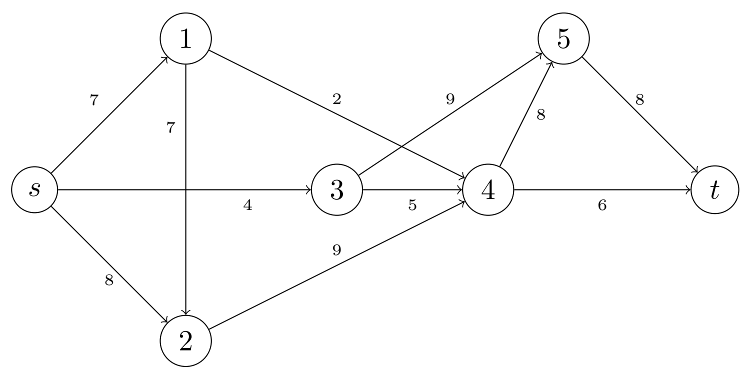
We get the following reduced costs for the nonbasic variables

$$x_{12} : u_1 + v_2 - c_{12} = 0 + 88 - 215 = -127$$
$$x_{31} : u_3 + v_1 - c_{31} = -20 + 80 - 102 = -42$$
$$x_{42} : u_4 + v_2 - c_{42} = 120 + 88 - 300 = -92$$

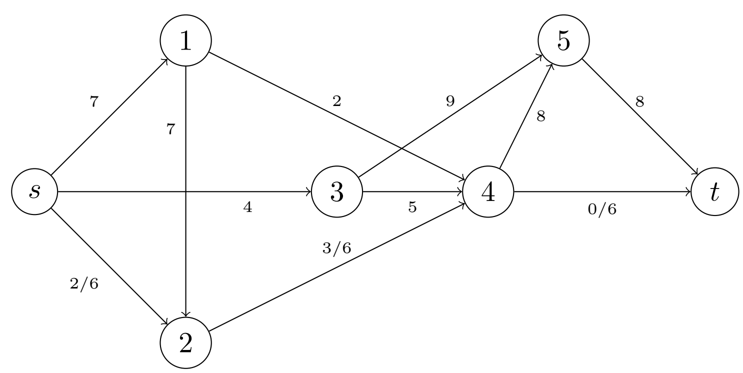
We conclude that this tableau is already optimal. So the minimum cost is $C = 80(1000) + 100(1100) + 108(200) + 68(1200) + 200(200) = \$333,200$ when the cars are shipped as shown above. In total we send Denver 1000 cars from Los Angeles and 1100 from Detroit (and fail to satisfy 200 cars' worth of demand in Denver) and send Miami 200 cars from Detroit and 1200 cars from New Orleans (and satisfy the whole demand in Miami).

Problem 7

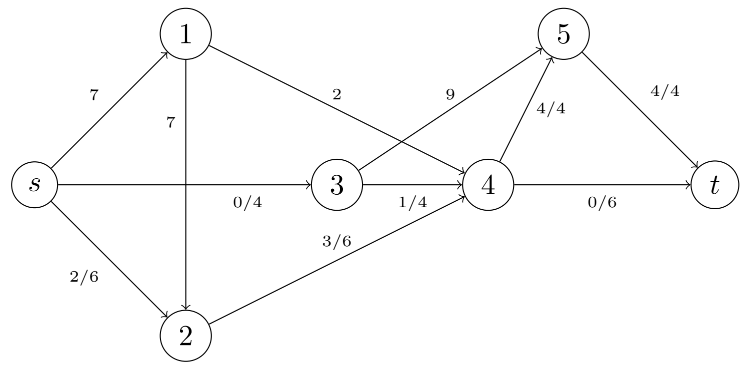
We begin with our original network, shown below.



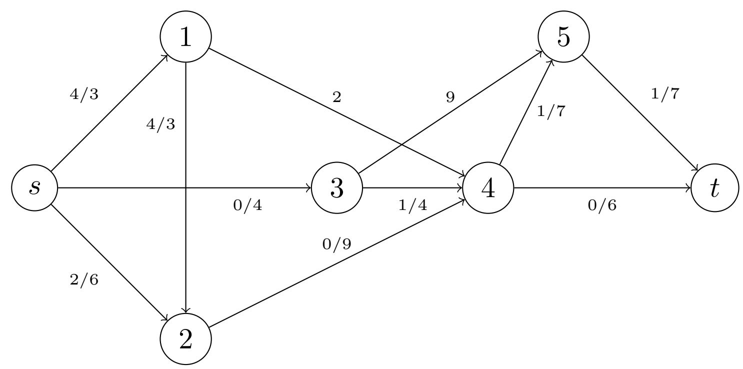
We notice that the path $s \to 2 \to 4 \to t$ has capacity 6. We start with it as our flow and get the following residual capacities.



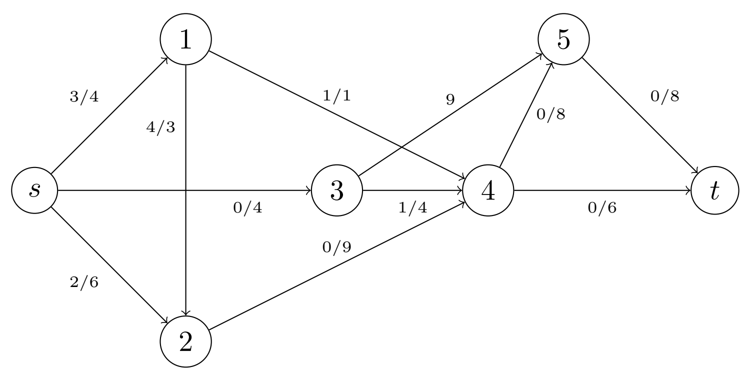
We see that the path $s \to 3 \to 4 \to 5 \to t$ has capacity 4. We add it to our flow, bringing us to 10, and relabel.



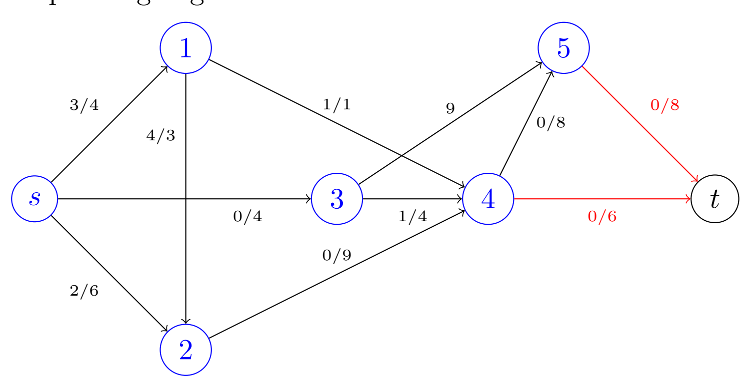
We see the path $s \to 1 \to 2 \to 4 \to 5 \to t$ has capacity 3; we add it, taking us to 13, and get the following residual network.



Finally the path $s \to 1 \to 4 \to t$ has capacity 1. We add it, getting to 14, and have the following residual network.

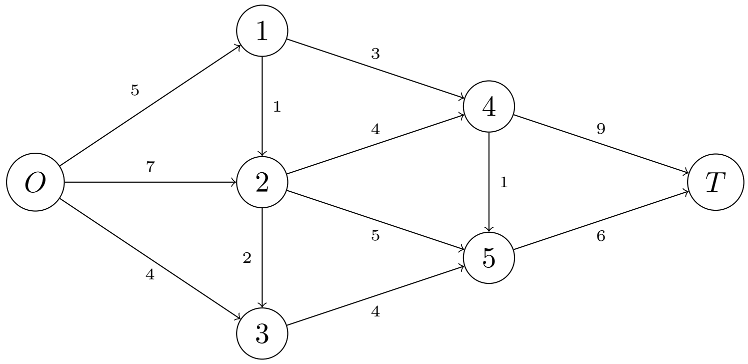


We are now done. The nodes reachable from the source along positive capacity paths are marked in blue and the corresponding edges in the minimal cut are marked in red.

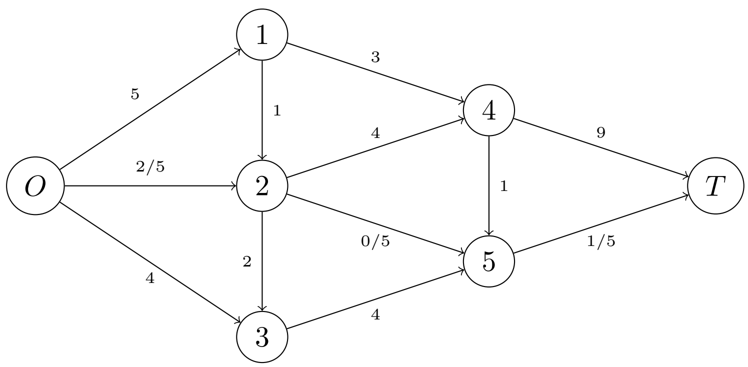


Problem 8

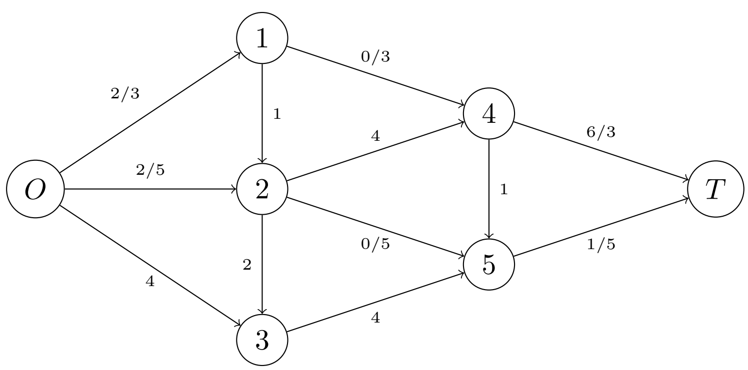
We start with our original network, shown below.



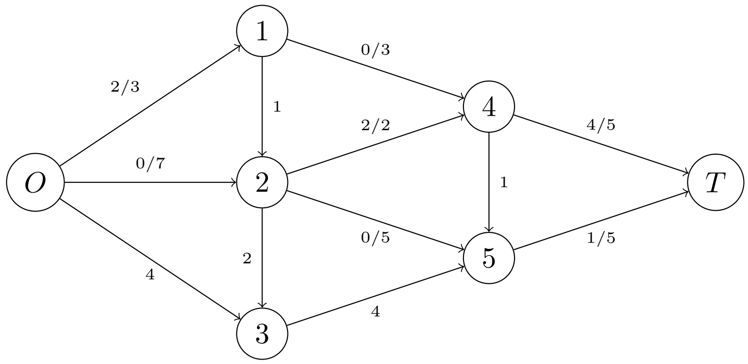
Now we look for paths with capacity. We notice that we can get a flow of 5 from the path $0 \to 2 \to 5 \to T$. The residual capacities are as follows.



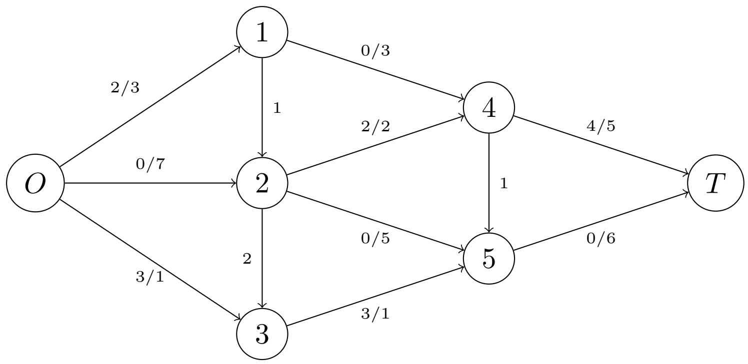
We notice that the path $O \to 1 \to 4 \to T$ has capacity 3. Adding that path to our flow we get the following residual capacities. Now we are up to a total flow of 8.



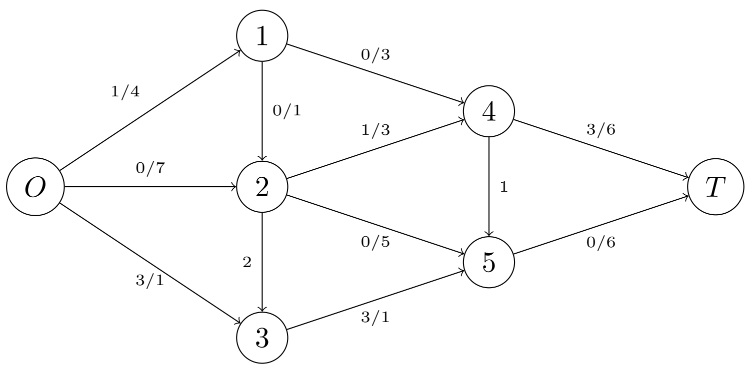
We perceive that the path $O \to 2 \to 4 \to T$ has capacity 2. We add it to our flow, getting up to a total flow of 10, and get the following residual capacities.



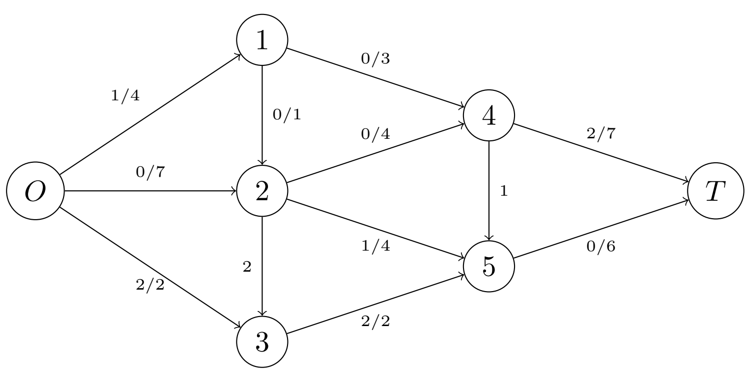
We see the path $O \to 3 \to 5 \to T$ has residual capacity 1 and add it in, taking us to 11.



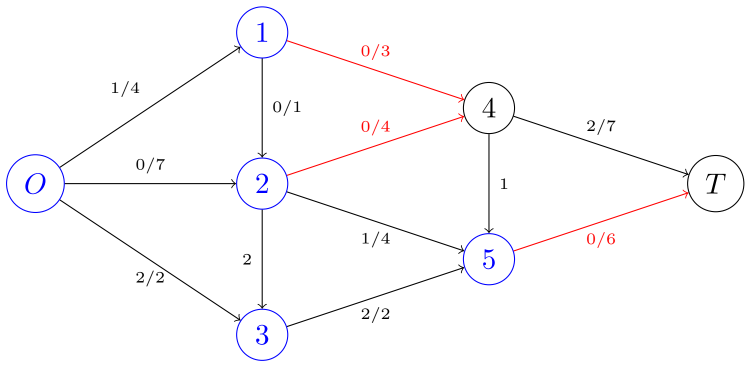
We note that the path $O \to 1 \to 2 \to 4 \to T$ has capacity 1. We add it in, taking us to 12.



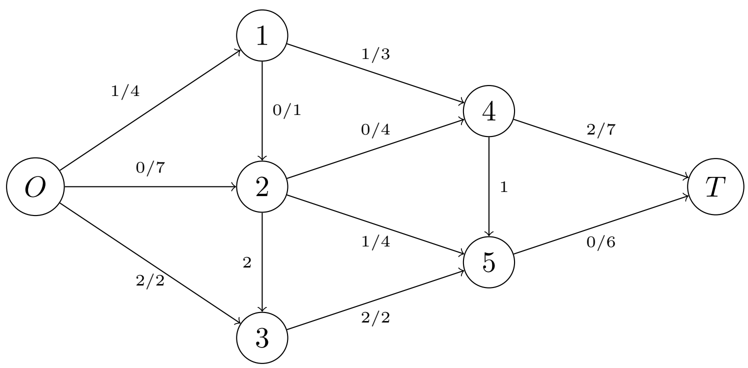
Finally, we may consider the path $O \to 3 \to 5 \to 2 \to 4 \to T$, which currently has capacity 1. (Here we are flowing backwards along the edge from 2 to 5 using the existing flow along the edge.) We add it in, taking us to 13, and have the following residual capacities.



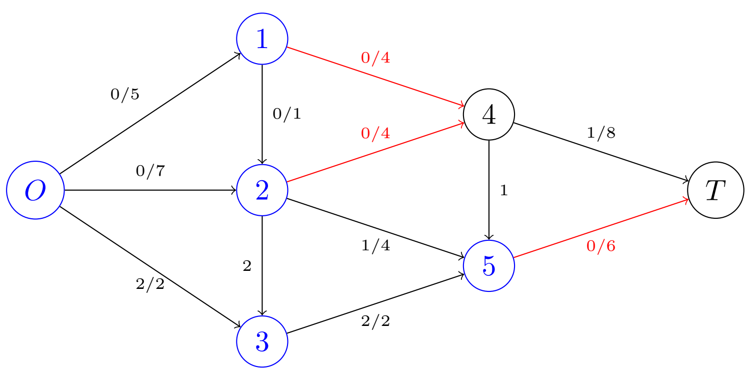
There are no longer any paths with nonzero capacity in the residual network. So the maximal flow is 13. The nodes reachable along a positive capacity path are marked in blue below and a minimum cut is marked in red.



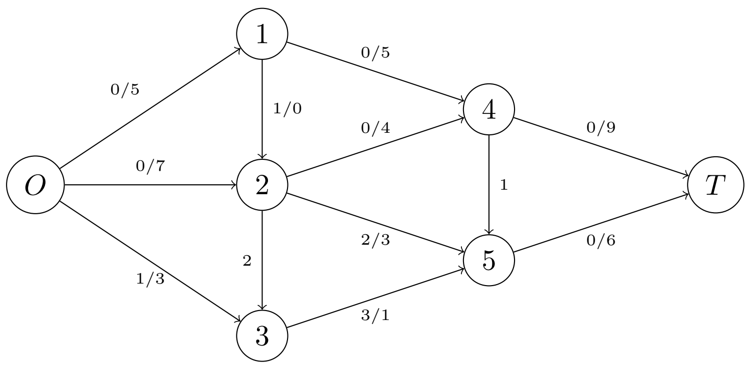
We consider our options for expanding the capacity by 2 units. Clearly we must change something in this minimum cut, so the cheapest expansion by 1 unit is to spend \$300 on expanding the line $1 \to 4$ by one unit. We add that in and consider our situation.



We see the path $O \to 1 \to 4 \to T$ has capacity 1. We add it to our flow and get the new residual network, with nodes reachable from the source by a positive capacity path marked in blue. The corresponding minimum cut is the same (and marked in red).



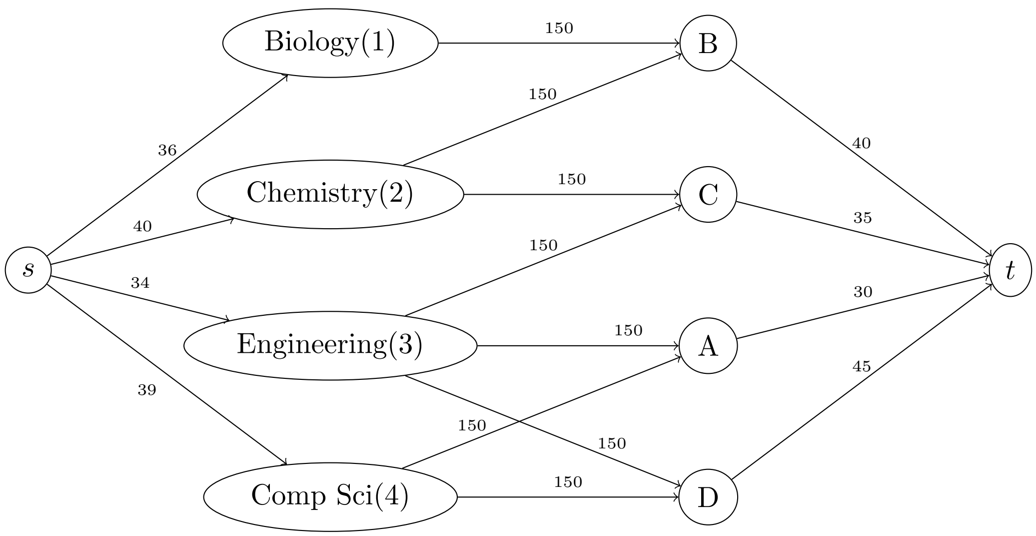
We see that we are still best off paying another \$300 to expand the line from $1 \to 4$ one more unit and using the new path $s \to 3 \to 5 \to 2 \to 1 \to T$. The residual capacities of the final network are shown below.



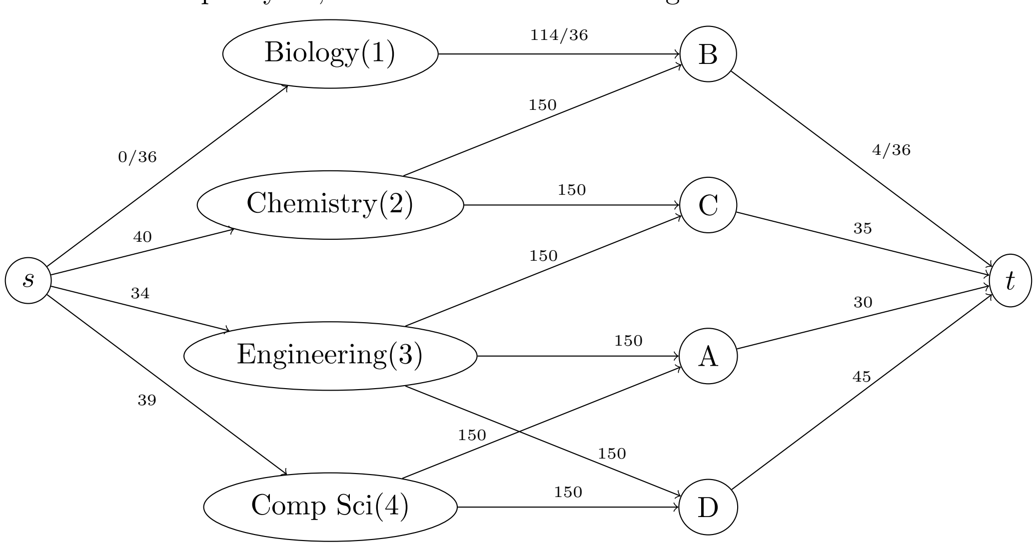
So we should expand the $1 \to 4$ pipeline by 2 units for a cost of \$600.

Problem 9

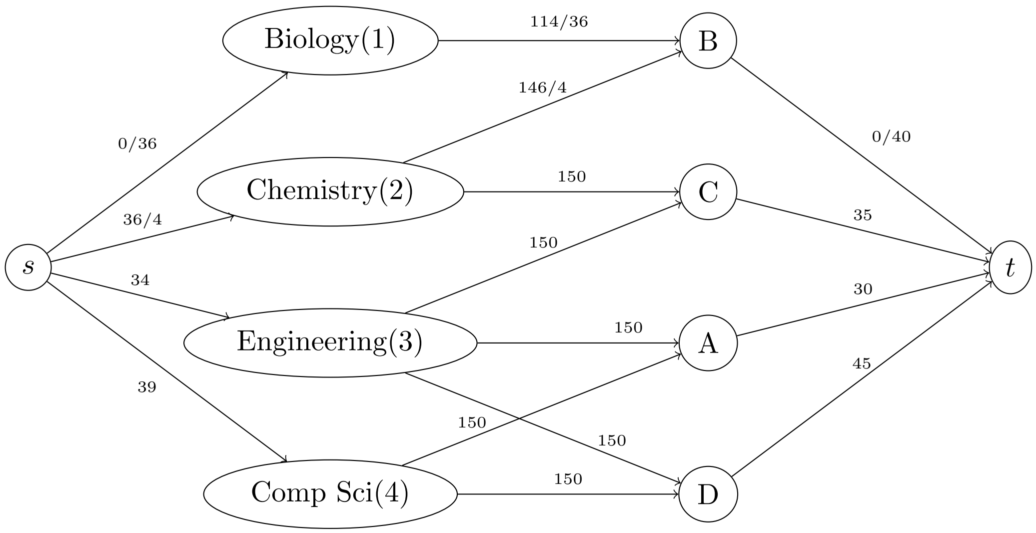
We can represent this as a network as shown.



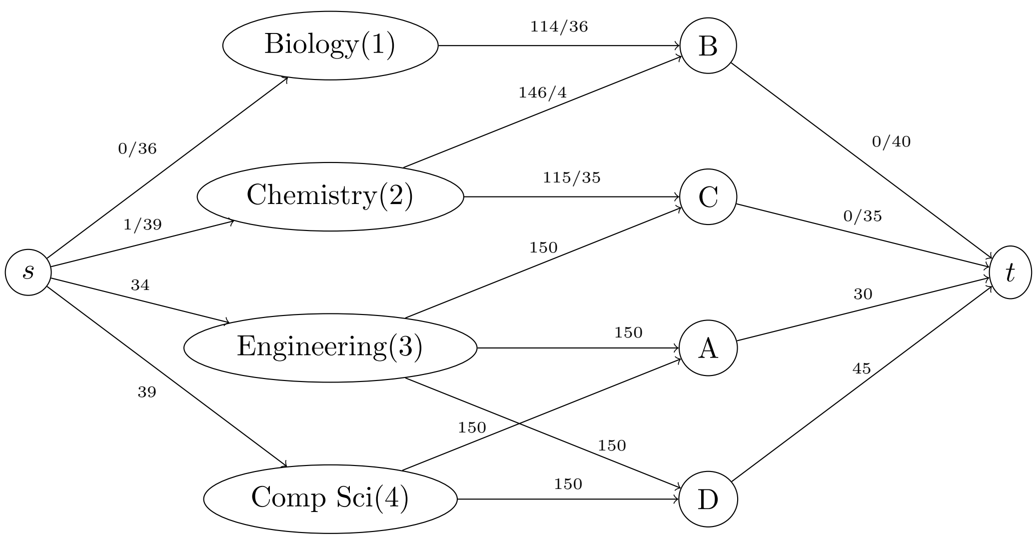
The capacities of 150 on the edges from majors to advisers could be replaced with ∞ for the same result. We want to know the maximum flow in this network. We first notice that the path $s \to 1 \to B \to t$ has capacity 36, and send 36 students along it.



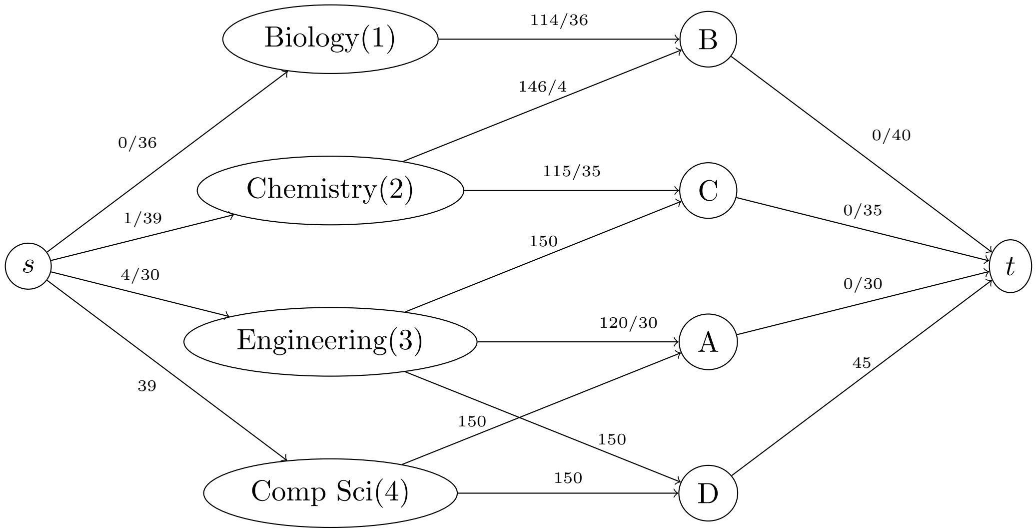
Now we see that $s \to 2 \to B \to t$ has capacity 4 and send 4 students along it.



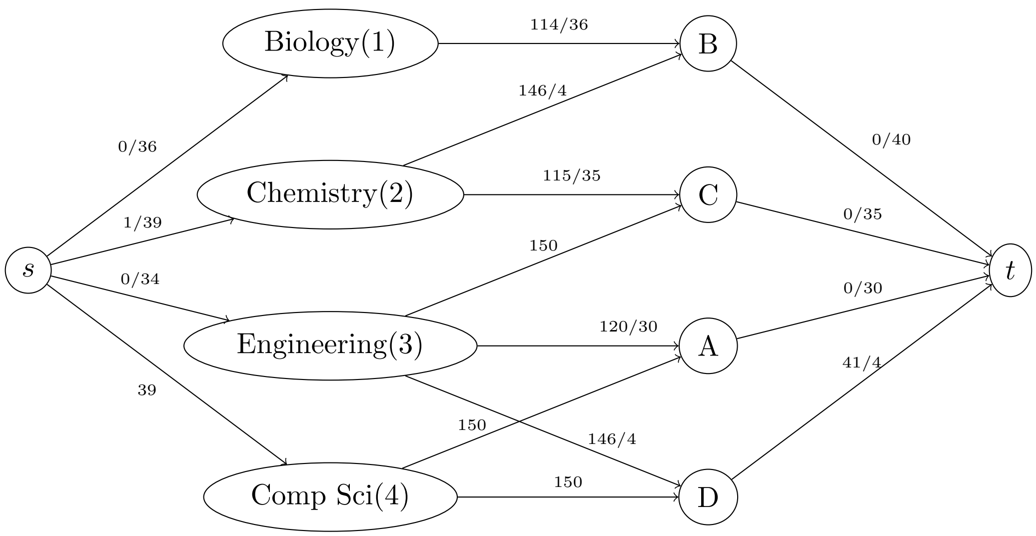
We further see that the path $s \to 2 \to C \to t$ has capacity 35 and send 35 students along it.



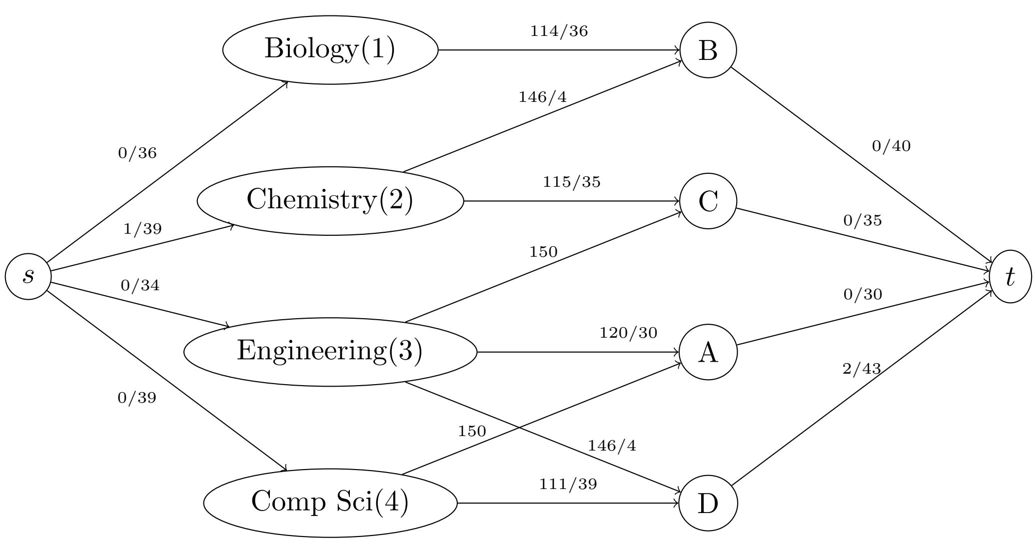
We now use the 30 student capacity of $s \to 3 \to A \to t$.



We now use the 4 student capacity of $s \to 3 \to D$.



Finally we can use the 39 student capacity of $s \to 4 \to D \to t$.



There aren't any further paths with capacity, so we stop. We see the maximum flow is 148; that is, 148 students can get an advising appointment this week.