Homework 12 Solutions

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Problem 3

We start with the cost matrix below.

$$C = \begin{bmatrix} 3 & 2 & 5 & 8 & 9 \\ 6 & 7 & 4 & 2 & 3 \\ 7 & 5 & 7 & 6 & 4 \\ 4 & 7 & 3 & 2 & 4 \\ 3 & 7 & 6 & 6 & 4 \end{bmatrix}$$

First, we subtract the smallest entry in each row from that row, resulting in the following.

$$\begin{bmatrix} 1 & 0 & 3 & 6 & 7 \\ 4 & 5 & 2 & 0 & 1 \\ 3 & 1 & 3 & 2 & 0 \\ 2 & 5 & 1 & 0 & 2 \\ 0 & 4 & 3 & 3 & 1 \end{bmatrix}$$

We now subtract the smallest entry in each column from that column, which only affects the third column in this case.

$$\begin{bmatrix} 1 & 0 & 2 & 6 & 7 \\ 4 & 5 & 1 & 0 & 1 \\ 3 & 1 & 2 & 2 & 0 \\ 2 & 5 & 0 & 0 & 2 \\ 0 & 4 & 2 & 3 & 1 \end{bmatrix}$$

We now attempt to perform an assignment by starring the first 0 in each row in an unoccupied column.

$$\begin{bmatrix} 1 & 0^* & 2 & 6 & 7 \\ 4 & 5 & 1 & 0^* & 1 \\ 3 & 1 & 2 & 2 & 0^* \\ 2 & 5 & 0^* & 0 & 2 \\ 0^* & 4 & 2 & 3 & 1 \end{bmatrix}$$

This produces a valid assignment, so we are done. On the original cost matrix C this assignment is

$$C = \begin{bmatrix} 3 & 2^* & 5 & 8 & 9 \\ 6 & 7 & 4 & 2^* & 3 \\ 7 & 5 & 7 & 6 & 4^* \\ 4 & 7 & 3^* & 2 & 4 \\ 3^* & 7 & 6 & 6 & 4 \end{bmatrix}$$

Hence the minimal cost is $2 + 2 + 4 + 3 + 3 = 14$.

Problem 4

We start with the cost matrix below.

$$C = \begin{bmatrix} 1 & 6 & 3 & 4 & 4 \\ 1 & 2 & 4 & 2 & 1 \\ 4 & 2 & 2 & 8 & 2 \\ 3 & 7 & 6 & 6 & 5 \\ 1 & 2 & 4 & 2 & 5 \end{bmatrix}$$

We start by subtracting the smallest value in each row from that row.

$$\begin{bmatrix} 0 & 5 & 2 & 3 & 3 \\ 0 & 1 & 3 & 1 & 0 \\ 2 & 0 & 0 & 6 & 0 \\ 0 & 4 & 3 & 3 & 2 \\ 0 & 1 & 3 & 1 & 4 \end{bmatrix}$$

Now we subtract the smallest value in each column from that column, which affects only the fourth column.

$$\begin{bmatrix} 0 & 5 & 2 & 2 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 2 & 0 & 0 & 5 & 0 \\ 0 & 4 & 3 & 2 & 2 \\ 0 & 1 & 3 & 0 & 4 \end{bmatrix}$$

We now attempt to make an assignment by starring the first 0 in each row in an unoccupied column.

$$\begin{bmatrix} 0^* & 5 & 2 & 2 & 3 \\ 0 & 1 & 3 & 0^* & 0 \\ 2 & 0^* & 0 & 5 & 0 \\ 0 & 4 & 3 & 2 & 2 \\ 0 & 1 & 3 & 0 & 4 \end{bmatrix}$$

We can only assign three rows this way. Now we look for zig-zag paths starting at 0's in unassigned rows that can improve this situation. First we check the 0 in the $(4,1)$ place in the matrix.

$$\begin{bmatrix} 0^* & 5 & 2 & 2 & 3 \\ 0 & 1 & 3 & 0^* & 0 \\ 2 & 0^* & 0 & 5 & 0 \\ 0 & 4 & 3 & 2 & 2 \\ 0 & 1 & 3 & 0 & 4 \end{bmatrix}$$

It has a path connecting it to the 0^* in the same column at $(1,1)$, but this path cannot be continued to a 0 in the first row. So, we mark the first column as necessary. The same is true of the 0 in the $(5,1)$ slot. However, if we look at the 0 in the $(5,4)$ slot, we see there is a path connecting it to the 0^* in the $(2,4)$ slot, which continues to the 0 in the $(2,5)$ slot.

$$\begin{bmatrix} 0^* & 5 & 2 & 2 & 3 \\ 0 & 1 & 3 & 0^* & 0 \\ 2 & 0^* & 0 & 5 & 0 \\ 0 & 4 & 3 & 2 & 2 \\ 0 & 1 & 3 & 0 & 4 \end{bmatrix}$$

As there is presently no 0^* in the fifth column we may switch the labels to get a new attempted assignment with four rows assigned.

$$\begin{bmatrix} 0^* & 5 & 2 & 2 & 3 \\ 0 & 1 & 3 & 0 & 0^* \\ 2 & 0^* & 0 & 5 & 0 \\ 0 & 4 & 3 & 2 & 2 \\ 0 & 1 & 3 & 0^* & 4 \end{bmatrix}$$

Now we cover the necessary column and the necessary rows (that is, the rows containing a 0^* not in the necessary column).

$$\begin{bmatrix} 0^* & 5 & 2 & 2 & 3 \\ 0 & 1 & 3 & 0 & 0^* \\ 2 & 0^* & 0 & 5 & 0 \\ 0 & 4 & 3 & 2 & 2 \\ 0 & 1 & 3 & 0^* & 4 \end{bmatrix}$$

We subtract the smallest uncovered entry, namely 2, from the uncovered entries and add it to the doubly covered entries.

$$\begin{bmatrix} 0 & 3 & 0 & 0 & 1 \\ 2 & 1 & 3 & 0 & 0 \\ 4 & 0 & 0 & 5 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 4 \end{bmatrix}$$

Now we attempt to perform an assignment.

$$\begin{bmatrix} 0^* & 3 & 0 & 0 & 1 \\ 2 & 1 & 3 & 0^* & 0 \\ 4 & 0^* & 0 & 5 & 0 \\ 0 & 2 & 1 & 0 & 0^* \\ 2 & 1 & 3 & 0 & 4 \end{bmatrix}$$

We can only assign four rows. But attempting to draw a path at the 0 in the unassigned row gives us a path terminating in a 0 which we may use to switch the labels and get a full assignment.

$$\begin{bmatrix} 0^* & 3 & 0 & 0 & 1 \\ 2 & 1 & 3 & 0^* & 0 \\ 4 & 0^* & 0 & 5 & 0 \\ 0 & 2 & 1 & 0 & 0^* \\ 2 & 1 & 3 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & 0^* & 0 & 1 \\ 2 & 1 & 3 & 0 & 0^* \\ 4 & 0^* & 0 & 5 & 0 \\ 0^* & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0^* & 4 \end{bmatrix}$$

We declare victory. On the original matrix this assignment is

$$C = \begin{bmatrix} 1 & 6 & 3^* & 4 & 4 \\ 1 & 2 & 4 & 2 & 1^* \\ 4 & 2^* & 2 & 8 & 2 \\ 3^* & 7 & 6 & 6 & 5 \\ 1 & 2 & 4 & 2^* & 5 \end{bmatrix}$$

So the minimal cost is $3+2+3+2+1=11$.

Problem 5

We start by forming a matrix of distances between the offices.

	706	707	710	714	715
702	4	5	8	12	13
705	1	2	5	9	10
708	2	1	2	6	7
709	3	2	1	5	6
713	7	6	3	1	2

We see that our cost matrix is

$$\begin{bmatrix} 4 & 5 & 8 & 12 & 13 \\ 1 & 2 & 5 & 9 & 10 \\ 2 & 1 & 2 & 6 & 7 \\ 3 & 2 & 1 & 5 & 6 \\ 7 & 6 & 3 & 1 & 2 \end{bmatrix}$$

We subtract the smallest entry in each row from that row.

$$\begin{bmatrix} 0 & 1 & 4 & 8 & 9 \\ 0 & 1 & 4 & 8 & 9 \\ 1 & 0 & 1 & 5 & 6 \\ 2 & 1 & 0 & 4 & 5 \\ 6 & 5 & 2 & 0 & 1 \end{bmatrix}$$

We further subtract the smallest entry in each column from that column, which only affects the final column.

$$\begin{bmatrix} 0 & 1 & 4 & 8 & 8 \\ 0 & 1 & 4 & 8 & 8 \\ 1 & 0 & 1 & 5 & 5 \\ 2 & 1 & 0 & 4 & 4 \\ 6 & 5 & 2 & 0 & 0 \end{bmatrix}$$

We now attempt to make an assignment by starring the first 0 in each row in an unoccupied column.

$$\begin{bmatrix} 0^* & 1 & 4 & 8 & 8 \\ 0 & 1 & 4 & 8 & 8 \\ 1 & 0^* & 1 & 5 & 5 \\ 2 & 1 & 0^* & 4 & 4 \\ 6 & 5 & 2 & 0^* & 0 \end{bmatrix}$$

This leaves the second row unassigned. The path from the 0 at $(2,1)$ up the first column to the 0^* in the $(1,1)$ slot cannot continue past that point, so the first column is necessary.

$$\begin{bmatrix} 0^* & 1 & 4 & 8 & 8 \\ 0 & 1 & 4 & 8 & 8 \\ 1 & 0^* & 1 & 5 & 5 \\ 2 & 1 & 0^* & 4 & 4 \\ 6 & 5 & 2 & 0^* & 0 \end{bmatrix}$$

We cover out the necessary column and the necessary rows (those with a 0^* outside the necessary column) with lines.

$$\begin{bmatrix} 0^* & 1 & 4 & 8 & 8 \\ 0 & 1 & 4 & 8 & 8 \\ 1 & 0^* & 1 & 5 & 5 \\ 2 & 1 & 0^* & 4 & 4 \\ 6 & 5 & 2 & 0^* & 0 \end{bmatrix}$$

Then we subtract the smallest uncovered entry, namely 1, from the uncovered entries and add it to the doubly-covered entries. We obtain the following.

$$\begin{bmatrix} 0 & 0 & 3 & 7 & 7 \\ 0 & 0 & 3 & 7 & 7 \\ 2 & 0 & 1 & 5 & 5 \\ 3 & 1 & 0 & 4 & 4 \\ 7 & 5 & 2 & 0 & 0 \end{bmatrix}$$

We again attempt to make an assignment.

$$\begin{bmatrix} 0^* & 0 & 3 & 7 & 7 \\ 0 & 0^* & 3 & 7 & 7 \\ 2 & 0 & 1 & 5 & 5 \\ 3 & 1 & 0^* & 4 & 4 \\ 7 & 5 & 2 & 0^* & 0 \end{bmatrix}$$

We can still only assign four rows. After checking the paths from the 0 in the unassigned row at the $(3,2)$ slot, we see that the first two columns are necessary, which makes the bottom two rows necessary.

$$\begin{bmatrix} 0^* & 0 & 3 & 7 & 7 \\ 0 & 0^* & 3 & 7 & 7 \\ 2 & 0 & 1 & 5 & 5 \\ 3 & 1 & 0^* & 4 & 4 \\ 7 & 5 & 2 & 0^* & 0 \end{bmatrix}$$

After subtracting the smallest uncovered entry, namely 1, from the uncovered entries and adding it to the doubly-covered entries, we obtain the following.

$$\begin{bmatrix} 0 & 0 & 2 & 6 & 6 \\ 0 & 0 & 2 & 6 & 6 \\ 2 & 0 & 0 & 4 & 4 \\ 4 & 2 & 0 & 4 & 4 \\ 8 & 6 & 2 & 0 & 0 \end{bmatrix}$$

We again attempt to make an assignment.

$$\begin{bmatrix} 0^* & 0 & 2 & 6 & 6 \\ 0 & 0^* & 2 & 6 & 6 \\ 2 & 0 & 0^* & 4 & 4 \\ 4 & 2 & 0 & 4 & 4 \\ 8 & 6 & 2 & 0^* & 0 \end{bmatrix}$$

We can still only assign four rows. After checking paths we see the first three columns are necessary (which makes only the fifth row necessary). We cover the necessary rows and columns.

$$\begin{bmatrix} 0^* & 0 & 2 & 6 & 6 \\ 0 & 0^* & 2 & 6 & 6 \\ 2 & 0 & 0^* & 4 & 4 \\ 4 & 2 & 0 & 4 & 4 \\ 8 & 6 & 2 & 0^* & 0 \end{bmatrix}$$

After subtracting 4 from the uncovered entries and adding it to the doubly-covered entries we have

$$\begin{bmatrix} 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 \\ 12 & 10 & 6 & 0 & 0 \end{bmatrix}$$

We attempt to perform an assignment.

$$\begin{bmatrix} 0^* & 0 & 2 & 2 & 2 \\ 0 & 0^* & 2 & 2 & 2 \\ 2 & 0 & 0^* & 0 & 0 \\ 4 & 2 & 0 & 0^* & 0 \\ 12 & 10 & 6 & 0 & 0^* \end{bmatrix}$$

Happily we are done! So we send the desks to new rooms as follows

	706	707	710	714	715
702	4*	5	8	12	13
705	1	2*	5	9	10
708	2	1	2*	6	7
709	3	2	1	5*	6
713	7	6	3	1	2*

for a total distance traveled of the length of 15 rooms.

Problem 6

We see that we could make this a minimization problem instead of a maximization problem by negating all of the entries of the matrix C. So we do that.

$$\begin{bmatrix} -3 & -2 & -1 & -4 & -5 \\ -1 & -5 & -9 & -2 & -6 \\ -6 & -3 & -7 & -5 & -5 \\ -2 & -8 & -5 & -3 & -5 \\ -1 & -5 & -7 & -6 & -5 \end{bmatrix}$$

The Hungarian method requires that our cost matrix have nonnegative entries. We won't change the best assignment if we add the absolute value of the entry of largest absolute value in each

row to that row, so we do that to get a matrix with nonnegative entries.

$$\begin{bmatrix} 2 & 3 & 4 & 1 & 0 \\ 8 & 4 & 0 & 7 & 3 \\ 1 & 4 & 0 & 2 & 2 \\ 6 & 0 & 3 & 5 & 3 \\ 6 & 2 & 0 & 1 & 2 \end{bmatrix}$$

Now we subtract the smallest entry in each column from that column.

$$\begin{bmatrix} 1 & 3 & 4 & 0 & 0 \\ 7 & 4 & 0 & 6 & 3 \\ 0 & 4 & 0 & 1 & 2 \\ 5 & 0 & 3 & 4 & 3 \\ 5 & 2 & 0 & 0 & 2 \end{bmatrix}$$

We now attempt to make an assignment.

$$\begin{bmatrix} 1 & 3 & 4 & 0^* & 0 \\ 7 & 4 & 0^* & 6 & 3 \\ 0^* & 4 & 0 & 1 & 2 \\ 5 & 0^* & 3 & 4 & 3 \\ 5 & 2 & 0 & 0 & 2 \end{bmatrix}$$

We are only able to assign four rows. However, we can easily see that there is a path from the 0 at the $(5,4)$ slot to the 0 at the $(1,4)$ slot to the 0 at the $(1,5)$ slot that fixes this.

$$\begin{bmatrix} 1 & 3 & 4 & 0^* & 0 \\ 7 & 4 & 0^* & 6 & 3 \\ 0^* & 4 & 0 & 1 & 2 \\ 5 & 0^* & 3 & 4 & 3 \\ 5 & 2 & 0 & 0 & 2 \end{bmatrix}$$

We change the assignment accordingly.

$$\begin{bmatrix} 1 & 3 & 4 & 0 & 0^* \\ 7 & 4 & 0^* & 6 & 3 \\ 0^* & 4 & 0 & 1 & 2 \\ 5 & 0^* & 3 & 4 & 3 \\ 5 & 2 & 0 & 0^* & 2 \end{bmatrix}$$

Now we have a complete assignment! On the original value matrix this assignment is

$$\begin{bmatrix} 3 & 2 & 1 & 4 & 5^* \\ 1 & 5 & 9^* & 2 & 6 \\ 6^* & 3 & 7 & 5 & 5 \\ 2 & 8^* & 5 & 3 & 5 \\ 1 & 5 & 7 & 6^* & 5 \end{bmatrix}$$

Hence the max value is $5 + 9 + 6 + 8 + 6 = 34$.