# Homework 11 Solutions

April 18, 2022

## Problem 3

We wish to add the constraint $x_1 + x_2 \geq 60$ to our original investment problem. We recall that the final tableau for this problem was

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|
| $u_4$ | 0 | 1 | 0 | 1 | 0 | $-1$ | 1 | 0 | 90 |
| $u_2$ | 0 | 2 | 0 | 1 | 1 | $-2$ | 0 | 0 | 120 |
| $x_1$ | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 40 |
| $x_3$ | 0 | 1 | 1 | 1 | 0 | $-1$ | 0 | 0 | 160 |
| | 0 | 10 | 0 | 50 | 0 | 40 | 0 | 1 | 11600 |

So we add a new row corresponding the the new constraint, written as $-x_1 - x_2 + u_5 = -60$.

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $u_4$ | 0 | 1 | 0 | 1 | 0 | $-1$ | 1 | 0 | 0 | 90 |
| $u_2$ | 0 | 2 | 0 | 1 | 1 | $-2$ | 0 | 0 | 0 | 120 |
| $x_1$ | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 40 |
| $x_3$ | 0 | 1 | 1 | 1 | 0 | $-1$ | 0 | 0 | 0 | 160 |
| $u_5$ | $-1$ | $-1$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | $-60$ |
| | 0 | 10 | 0 | 50 | 0 | 40 | 0 | 0 | 1 | 11600 |

We need to clear out the new row to get a tableau, as follows.

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $u_4$ | 0 | 1 | 0 | 1 | 0 | $-1$ | 1 | 0 | 0 | 90 |
| $u_2$ | 0 | 2 | 0 | 1 | 1 | $-2$ | 0 | 0 | 0 | 120 |
| $x_1$ | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 40 |
| $x_3$ | 0 | 1 | 1 | 1 | 0 | $-1$ | 0 | 0 | 0 | 160 |
| $u_5$ | 0 | $-1$ | 0 | 0 | 0 | 1 | 0 | 1 | 0 | $-20$ |
| | 0 | 10 | 0 | 50 | 0 | 40 | 0 | 0 | 1 | 11600 |

Now we attempt to pivot back to feasibility. We let $u_5$ be the departing variable and $x_2$ be the entering variable. This gives us the following.

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $u_4$ | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 70 |
| $u_2$ | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 2 | 0 | 80 |
| $x_1$ | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 40 |
| $x_3$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 140 |
| $x_2$ | 0 | 1 | 0 | 0 | 0 | $-1$ | 0 | $-1$ | 0 | 20 |
| | 0 | 0 | 0 | 50 | 0 | 50 | 0 | 0 | 1 | 11400 |

So \$40,000 should be invested in the utilities stock, \$20,000 should be invested in the electronics stock, and \$140,000 should be invested in the bond, for an annual return of \$11,400.

## Problem 4

We wish to mazimize $z = -22x_1 - 36x_2$ subject to the constraints

$$\begin{cases} 50x_1 + 140x_2 \leq 350 \\ 100x_1 + 200x_2 \geq 600 \\ x_1, x_2 \leq 0 \end{cases}$$

We may as well remove common factors from these equations, since we're not expecting to need to solve the dual problem. We do that and add slack and artificial variables as follows.

$$\begin{cases} 5x_1 + 14x_2 + u_1 = 35 \\ x_1 + 2x_2 - u_2 + y_1 = 6 \\ x_1, x_2, u_1, u_2, y_1 \leq 0 \end{cases}$$

Now we can start Phase I, maximizing $z = -y_1$. We begin with the following.

| | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $y_1$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $u_1$ | 5 | 14 | 1 | 0 | 0 | 0 | 35 |
| $y_1$ | 1 | 2 | 0 | $-1$ | 1 | 0 | 6 |
| | 0 | 0 | 0 | 0 | 1 | 1 | 0 |

We clear out the objective row to get our intial tableau.

| | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $y_1$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $u_1$ | 5 | 14 | 1 | 0 | 0 | 0 | 35 |
| $y_1$ | 1 | 2 | 0 | $-1$ | 1 | 0 | 6 |
| | $-1$ | $-2$ | 0 | 1 | 0 | 1 | $-6$ |

Now to pivot. We notice that if we use Bland's Rule here and pick $x_1$ to be the entering variable we can let $y_1$ be the departing variable to end Phase I, so we do that. After pivoting we have the following.

| | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $y_1$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $u_1$ | 0 | 4 | 1 | 5 | $-5$ | 0 | 5 |
| $x_1$ | 1 | 2 | 0 | $-1$ | 1 | 0 | 6 |
| | 0 | 0 | 0 | 0 | 1 | 1 | 0 |

Now we drop the column for the artificial variable and restore the original objective function.

| | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|
| $u_1$ | 0 | 4 | 1 | 5 | 0 | 5 |
| $x_1$ | 1 | 2 | 0 | $-1$ | 0 | 6 |
| | 22 | 36 | 0 | 0 | 1 | 0 |

We then clear out the objective row.

| | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|
| $u_1$ | 0 | 4 | 1 | 5 | 0 | 5 |
| $x_1$ | 1 | 2 | 0 | $-1$ | 0 | 6 |
| | 0 | $-8$ | 0 | 22 | 1 | $-132$ |

Now we have $x_2$ entering and $u_1$ departing. We pivot.

| | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 1/4 | 5/4 | 0 | 5/4 |
| $x_1$ | 1 | 0 | $-1/2$ | $-7/2$ | 0 | 7/2 |
| | 0 | 0 | 2 | 32 | 1 | $-122$ |

We have now finished the simplex method. Currently we have a best value of $z = -122$ with $(x_1, x_2) = (7/2, 5/4)$. We branch using $x_1 \leq 3$ and $x_1 \geq 4$. When we add $x_1 \leq 3$ to the tableau as $x_1 + u_3 = 3$ we get the following.

| | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $u_3$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 1/4 | 5/4 | 0 | 0 | 5/4 |
| $x_1$ | 1 | 0 | $-1/2$ | $-7/2$ | 0 | 0 | 7/2 |
| $u_3$ | 1 | 0 | 0 | 0 | 1 | 0 | 3 |
| | 0 | 0 | 2 | 32 | 0 | 1 | $-122$ |

We clear out the $u_3$ row and obtain the following tableau

| | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $u_3$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 1/4 | 5/4 | 0 | 0 | 5/4 |
| $x_1$ | 1 | 0 | $-1/2$ | $-7/2$ | 0 | 0 | 7/2 |
| $u_3$ | 0 | 0 | 1/2 | 7/2 | 1 | 0 | $-1/2$ |
| | 0 | 0 | 2 | 32 | 0 | 1 | $-122$ |

There is no way to pivot this back to feasibility! So, there are no solutions to this linear programming problem with $x_1 \leq 3$. Along the other branch, when we add $x_1 \geq 4$ to the tableau as $-x_1 + u_3 = -4$ we get the following

| | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $u_3$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 1/4 | 5/4 | 0 | 0 | 5/4 |
| $x_1$ | 1 | 0 | $-1/2$ | $-7/2$ | 0 | 0 | 7/2 |
| $u_3$ | $-1$ | 0 | 0 | 0 | 1 | 0 | $-4$ |
| | 0 | 0 | 2 | 32 | 0 | 1 | $-122$ |

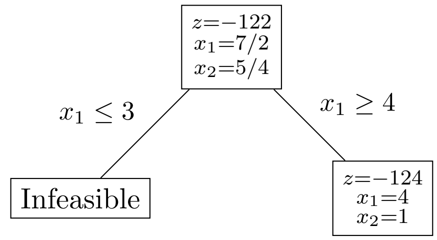
We clear out the $u_3$ row and obtain the following tableau.

| | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $u_3$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 1/4 | 5/4 | 0 | 0 | 5/4 |
| $x_1$ | 1 | 0 | $-1/2$ | $-7/2$ | 0 | 0 | 7/2 |
| $u_3$ | 0 | 0 | $-1/2$ | $-7/2$ | 1 | 0 | $-1/2$ |
| | 0 | 0 | 2 | 32 | 0 | 1 | $-122$ |

We let $u_3$ be the departing variable and $u_1$ be the entering variable. We do a dual pivot and obtain the following tableau.

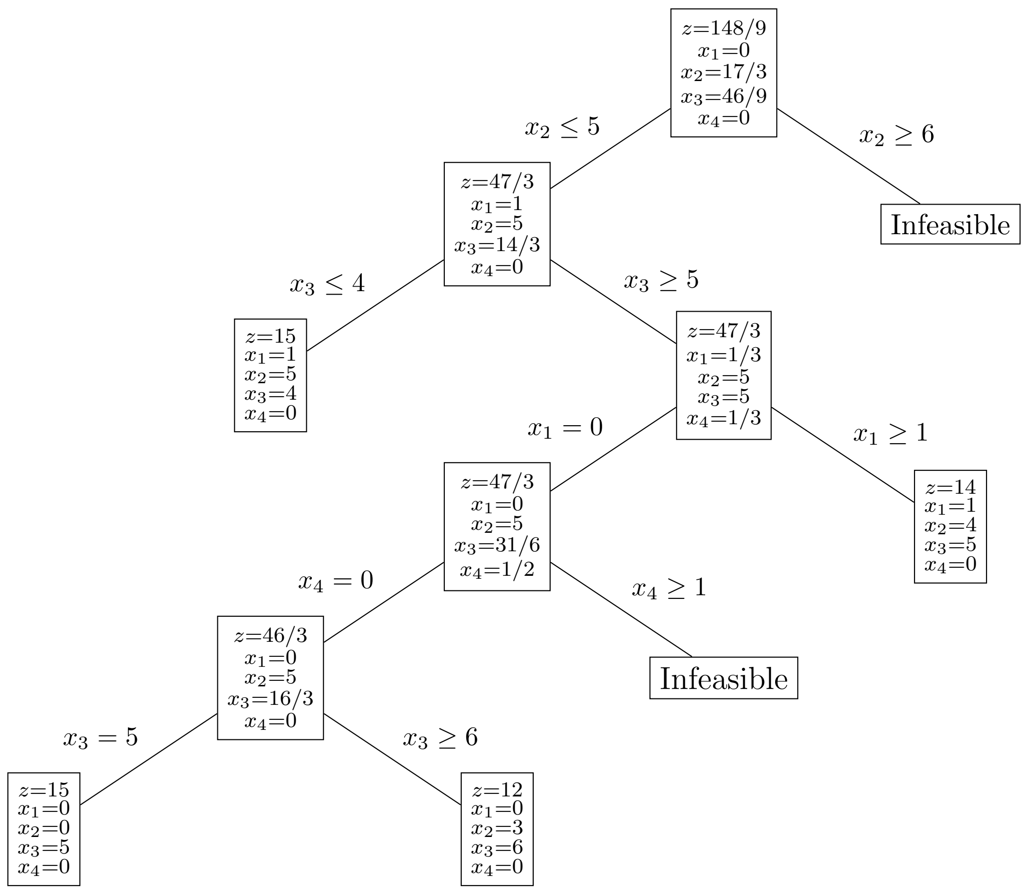
| | $x_1$ | $x_2$ | $u_1$ | $u_2$ | $u_3$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | 0 | $-1/2$ | 1/2 | 0 | 1 |
| $x_1$ | 1 | 0 | 0 | 0 | 1 | 0 | 4 |
| $u_1$ | 0 | 0 | 1 | 7 | $-2$ | 0 | 1 |
| | 0 | 0 | 0 | 18 | 4 | 1 | $-124$ |

We see in this branch the optimal solution is $-124$, or a cost of \$124,000, at $(x_1, x_2) = (4, 1)$. There are no further dangling branches, so we are done.




## Problem 5

Our tree is below.




We see that the maximum value of $z$ at an integer solution is 15 and it occurs at multiple points including the two in the tree above. Answers that point out that once you have an integer solution of 15, then given that all of the coefficients of the objective function are integers, you can ignore any node with a current $z$ value of less than 16 are also correct.