Homework 10 Solutions

April 7, 2022

Problem 4

Let the number of ounces of walnuts used be x_1, the number of ounces of pecans used be x_2, and the number of ounces of almonds used be x_3. The primal problem is to minimize $z = 12x_1 + 7x_2 + 6x_3$ subject to the constraints

$$\begin{cases} 6x_1 + x_2 + 2x_3 \geq 24 \\ 3x_1 + 3x_2 + x_3 \geq 22 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

to which we may add slack variables

$$\begin{cases} 6x_1 + x_2 + 2x_3 - t_1 = 24 \\ 3x_1 + 3x_2 + x_3 - t_2 = 22 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

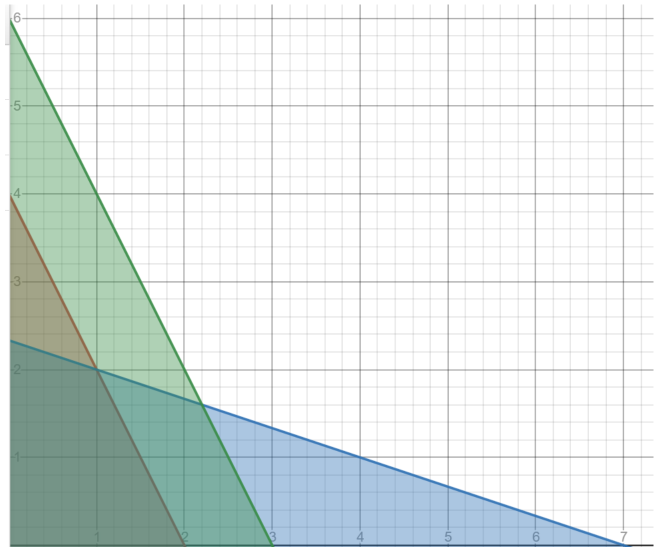
The dual problem then is to maximize $z' = 24x_1 + 22x_2$ subject to

$$\begin{cases} 6w_1 + 3w_2 \leq 12 \\ w_1 + 3w_2 \leq 7 \\ 2w_1 + w_2 \leq 6 \\ w_1, w_2 \geq 0 \end{cases}$$

To which we may add slack variables

$$\begin{cases} 6w_1 + 3w_2 + u_1 = 12 \\ w_1 + 3w_2 + u_2 = 7 \\ 2w_1 + w_2 + u_3 = 6 \\ w_1, w_2 \geq 0 \end{cases}$$

We graph the standard form of the dual problem, as below.



We notice that the extreme points of the region of feasible solutions occur at $(0, 0)$, $(2, 0)$, $(0, 7/3)$, and $(1, 2)$. We observe that the maximal value of z' on these values is $z' = 68$ at $(w_1, w_2) = (1, 2)$. We see that at this optimal solution we have $u_1 = u_2 = 0$ and $u_3 > 0$. So, in the optimal solution to the primal problem we certainly have $x_3 = 0$, since $x_3 u_3 = 0$ by complementary slackness. Now, since $w_1 t_1 = w_2 t_2 = 0$ at the optimal solution as well, we see that $t_1 = t_2 = 0$. So, at the optimal solution to the primal problem we must have

$$\begin{cases} 6x_1 + x_2 = 24 \\ 3x_1 + 3x_2 = 22 \end{cases}$$

The solution to this is $x_1 = 10/3$ and $x_2 = 4$. So, the optimal solution to the primal problem is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 4 \\ 0 \end{bmatrix}$$

So, the store would make a mix with 10/3 ounces of walnuts and 4 ounces of pecans for a total cost of 68 cents.

Problem 5

We recall that the primal problem was to maximize $z = 10x_1 + 12x_2$ subject to the constraints

$$\begin{cases} 3x_1 + 3x_2 \leq 240 \\ 3x_1 + 4x_2 \leq 240 \\ 3x_1 + 2x_2 \leq 180 \\ x_1, x_2 \geq 0 \end{cases}$$

where x_1 represents the number of kilos of chili-flavored chips made, x_2 is the number of kilos of pizza-flavored chips made, and the three constraints are the availability of the fryer, flavorer,

and packer in that order. The final tableau for this problem is

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	$-1/2$	0	30
x_2	0	1	0	$1/2$	$-1/2$	0	30
x_1	1	0	0	$-1/3$	$2/3$	0	40
	0	0	0	$8/3$	$2/3$	1	760

Now we can answer some questions.

(a) Suppose the profit of chili-flavored chips falls to 8 cents per kilogram, which is 2 less than the original profit of 10. This gives us the following

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	$-1/2$	0	30
x_2	0	1	0	$1/2$	$-1/2$	0	30
x_1	1	0	0	$-1/3$	$2/3$	0	40
	2	0	0	$8/3$	$2/3$	1	760

We must then clear out the new entry in the objective row for this to be a tableau:

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	$-1/2$	0	30
x_2	0	1	0	$1/2$	$-1/2$	0	30
x_1	1	0	0	$-1/3$	$2/3$	0	40
	0	0	0	$10/3$	$-2/3$	1	680

We observe this is *not* still optimal. So, the manufacturer should change what distribution of chips is being made. To find out how, we pivot with entering variable u_3 and departing variable x_1, obtaining

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	0	0	60
x_2	0	1	0	$1/2$	0	0	60
u_3	$3/2$	0	0	$-1/2$	1	0	60
	1	0	0	3	0	1	720

So in this case the manufacturer should make 60 kilos of pizza-flavored chips for a total profit of \$7.20. Next, suppose the profit of chili-flavored chips rises to 12 cents per kilogram, which is 2 more than the original profit of 10 cents per kilogram. This gives us the following

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	$-1/2$	0	30
x_2	0	1	0	$1/2$	$-1/2$	0	30
x_1	1	0	0	$-1/3$	$2/3$	0	40
	-2	0	0	$8/3$	$2/3$	1	760

We clear out the objective row to get a tableau:

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	$-1/2$	0	30
x_2	0	1	0	$1/2$	$-1/2$	0	30
x_1	1	0	0	$-1/3$	$2/3$	0	40
	0	0	0	2	2	1	840

This tableau is still optimal! So, the manufacturer should still make 40 kilos of chili-flavored chips and 30 kilos of pizza-flavored chips, with a new profit of \$8.40.

(b) Suppose that the new profit on pizza-flavored chips is $12 + \Delta c_2$. We consider the resulting change to the final tableau:

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	$-1/2$	0	30
x_2	0	1	0	$1/2$	$-1/2$	0	30
x_1	1	0	0	$-1/3$	$2/3$	0	40
	0	$-\Delta c_2$	0	$8/3$	$2/3$	1	760

We clear out the objective row to obtain:

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	$-1/2$	0	30
x_2	0	1	0	$1/2$	$-1/2$	0	30
x_1	1	0	0	$-1/3$	$2/3$	0	40
	0	0	0	$8/3 + 1/2(\Delta c_2)$	$2/3 - 1/2(\Delta c_2)$	1	$760 + 30\Delta c_2$

This is optimal exactly if we have $8/3 + 1/2(\Delta c_2) \geq 0$ and $2/3 - 1/2(\Delta c_2) \geq 0$, so if $-4/3 \leq \Delta c_2 \leq 1/3$. Thus the price of pizza-flavored chips can fall by as much as 4/3 cents per kilogram, to 33/3 cents and rise by as much as 1/3 cents per kilogram to 37/3 cents per kilogram before the manufacturer should change the distribution of chips made.

(c) We recall that the availability of the packer is the third constraint in our list of constraints, and it is currently available for three hours per day. If it's availability rises by half an hour, we add 30 times the column under the third slack variable u_3 to the final column of our tableau to obtain

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	$-1/2$	0	15
x_2	0	1	0	$1/2$	$-1/2$	0	15
x_1	1	0	0	$-1/3$	$2/3$	0	60
	0	0	0	$8/3$	$2/3$	1	780

This solution is still feasible, so the manufacturer should make 60 kilos of chili-flavored chips and 15 kilos of pizza-flavored chips for a total profit of \$7.80.

(d) We recall the the availability of the flavorer is the second constraint in our list of constraints above, and it is currently available for four hours per day. If its availability falls to two hours per day we should add -120 times the column under u_2 to the final column

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	$-1/2$	0	90
x_2	0	1	0	$1/2$	$-1/2$	0	-30
x_1	1	0	0	$-1/3$	$2/3$	0	80
	0	0	0	$8/3$	$2/3$	1	440

This is no longer feasible, so we need to pivot to restore feasibility. We select x_2 for the departing variable and u_3 for the entering variable, since it has the only negative entry in the pivotal row.

We then pivot and obtain

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	-1	1	-1	0	0	120
u_3	0	-2	0	-1	1	0	60
x_1	1	$4/3$	0	$1/3$	0	0	40
	0	$4/3$	0	$10/3$	0	1	400

So, in this case the manufacturer should make 40 kilograms of chili-flavored chips, for a total profit of \$4.00.

Problem 6

Recall that the problem is to maximize $z = x_1 + 3x_3$ subject to the constraints

$$\begin{cases} x_1 + 2x_2 + 4x_3 = 4 \\ x_1 + 3x_2 + 2x_3 = 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

and that the final tableau was

	x_1	x_2	x_3	z	
x_1	1	0	8	0	2
x_2	0	1	-2	0	1
	0	0	5	1	2

suppose we replace c_1 with $c_1' = c_1 + \Delta c_1$. Then the tableau becomes

	x_1	x_2	x_3	z	
x_1	1	0	8	0	2
x_2	0	1	-2	0	1
	$-\Delta c_1$	0	5	1	2

and we clear out the objective row to obtain

	x_1	x_2	x_3	z	
x_1	1	0	8	0	2
x_2	0	1	-2	0	1
	0	0	$5 + 8\Delta c_1$	1	$2 + 2\Delta c_1$

We see this is still optimal as long as $5 + 8\Delta c_1 \geq 0$, or $\Delta c_1 \geq -5/8$, meaning that $c_1' \geq 3/8$. Next, suppose we replace c_3 with $c_3' = c_3 + \Delta c_3$. Then the tableau becomes

	x_1	x_2	x_3	z	
x_1	1	0	8	0	2
x_2	0	1	-2	0	1
	0	0	$5 - \Delta c_3$	1	2

We see that this stays optimal as long as $\Delta c_3 \leq 5$, or $c_3' \leq 8$.