Math 354, Section 04
Linear Optimization

Sample Final

Instructions: You have three hours to complete the exam. There are six questions, worth a total of 60 points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ____________________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. Consider the following linear programming problem: Maximize $z = 3x_1 + 2x_2 - x_3 + 3x_4$ subject to the constraints

$$\begin{cases} 2x_1 + x_2 + 2x_3 \leq 16 \\ 3x_1 + x_2 + x_4 \leq 30 \\ 5x_1 + x_2 + 3x_3 + 2x_4 \leq 35 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

The final tableau for the simplex method applied to this problem is shown below.

	x_1	x_2	x_3	x_4	u_1	u_2	u_3	z	
x_2	2	1	2	0	1	0	0	0	16
u_2	$-1/2$	0	$-5/2$	0	$-1/2$	1	$-1/2$	0	$9/2$
x_4	$3/2$	0	$1/2$	1	$-1/2$	0	$1/2$	0	$19/2$
	$11/2$	0	$13/2$	0	$1/2$	0	$3/2$	1	$121/2$

(a) [1pts.] What basic feasible solution does this tableau represent?

Solution: The tableau represents the basic feasible solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \\ 0 \\ 19/2 \\ 0 \\ 9/2 \\ 0 \end{bmatrix}.$$

The same thing with the slack variables dropped is also fine.

(b) [2pts.] What is the dual problem to the linear programming problem above, and what is its optimal solution?

Solution: The dual problem is to minimize $z' = 16w_1 + 30w_2 + 35w_3$ subject to the constraints

$$\begin{cases} 2w_1 + 3w_2 + 5w_3 \geq 3 \\ w_1 + w_2 + w_3 \geq 2 \\ 2w_1 + 3w_3 \geq -1 \\ w_2 + 2w_3 \geq 3 \\ w_1, w_2, w_3 \geq 0 \end{cases}$$

The optimal value is the same as the primal problem, $121/2$. Looking at the entries in the tableau above under the slack variables, we see that the optimal

solution is

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 3/2 \end{bmatrix}.$$

(c) [7pts.] What is the best solution to this problem if we add the constraint that x_1, x_2, x_3, x_4 are integers?

Solution: We may employ a branch and bound strategy. Notice that $121/2 = 60.5$ is an upper bound on the optimal value we are looking for; indeed, since the coefficients of the original objective function are integers we see that we actually have an upper bound of 60. We currently currently have $x_4 = 19/2 = 9.5$; we can consider the sets of solutions with $x_4 \leq 9$ and $x_4 \geq 10$. We start with the first one, adding in the constraint $x_4 + u_4 = 9$.

	x_1	x_2	x_3	x_4	u_1	u_2	u_3	u_4	z	
x_2	2	1	2	0	1	0	0	0	0	16
u_2	$-1/2$	0	$-5/2$	0	$-1/2$	1	$-1/2$	0	0	$9/2$
x_4	$3/2$	0	$1/2$	1	$-1/2$	0	$1/2$	0	0	$19/2$
u_4	0	0	0	1	0	0	0	1	0	9
	$11/2$	0	$13/2$	0	$1/2$	0	$3/2$	0	1	$121/2$

We need to remove the 1 in the u_4 row to have a tableau, so we subtract the x_4 row to clear it out.

	x_1	x_2	x_3	x_4	u_1	u_2	u_3	u_4	z	
x_2	2	1	2	0	1	0	0	0	0	16
u_2	$-1/2$	0	$-5/2$	0	$-1/2$	1	$-1/2$	0	0	$9/2$
x_4	$3/2$	0	$1/2$	1	$-1/2$	0	$1/2$	0	0	$19/2$
u_4	$-3/2$	0	$-1/2$	0	$1/2$	0	$-1/2$	1	0	$-1/2$
	$11/2$	0	$13/2$	0	$1/2$	0	$3/2$	0	1	$121/2$

This is no longer feasible, so we must do a dual pivot to restore feasibility. The departing variable is u_4 and the entering variable is u_3. We pivot.

	x_1	x_2	x_3	x_4	u_1	u_2	u_3	u_4	z	
x_2	2	1	2	0	1	0	0	0	0	16
u_2	1	0	-2	0	-1	1	0	-1	0	5
x_4	0	0	0	1	0	0	0	1	0	9
u_3	3	0	1	0	-1	0	1	-2	0	1
	1	0	5	0	2	0	0	0	1	59

This gives us an integer solution with $z = 59$. We now investigate the case

$x_4 \geq 10$ by adding $-x_4 + u_4 = -10$ to the original final tableau.

	x_1	x_2	x_3	x_4	u_1	u_2	u_3	u_4	z	
x_2	2	1	2	0	1	0	0	0	0	16
u_2	$-1/2$	0	$-5/2$	0	$-1/2$	1	$-1/2$	0	0	$9/2$
x_4	$3/2$	0	$1/2$	1	$-1/2$	0	$1/2$	0	0	$19/2$
u_4	0	0	0	-1	0	0	0	1	0	-10
	$11/2$	0	$13/2$	0	$1/2$	0	$3/2$	0	1	$121/2$

Again we must clear out the x_4 column to get a tableau.

	x_1	x_2	x_3	x_4	u_1	u_2	u_3	u_4	z	
x_2	2	1	2	0	1	0	0	0	0	16
u_2	$-1/2$	0	$-5/2$	0	$-1/2$	1	$-1/2$	0	0	$9/2$
x_4	$3/2$	0	$1/2$	1	$-1/2$	0	$1/2$	0	0	$19/2$
u_4	$3/2$	0	$1/2$	0	$-1/2$	0	$1/2$	1	0	$-1/2$
	$11/2$	0	$13/2$	0	$1/2$	0	$3/2$	0	1	$121/2$

Now we attempt to pivot back to feasibility. The departing variable is u_4, which makes the entering variable u_1. We pivot.

	x_1	x_2	x_3	x_4	u_1	u_2	u_3	u_4	z	
x_2	5	1	3	0	0	0	1	2	0	15
u_2	-2	0	-3	0	0	1	-1	-1	0	5
x_4	0	0	0	1	0	0	0	-1	0	10
u_1	-3	0	-1	0	1	0	-1	-2	0	1
	7	0	7	0	0	2	1	0	1	60

This must be the best integer solution (even without the information that we also reached a smaller integer solution on the other branch). So, the best value taken by this problem at an integer solution is 60 and it occurs at

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \\ 0 \\ 10 \end{bmatrix}.$$

Remark: The approximately corresponding problem on the actual final has slightly easier algebra.

2. Rutgers landscaping is purchasing plants for campus, and needs 350 plants with red flowers and 300 plants with white flowers. A flat of poppies contains 15 plants with red flowers and 20 plants with white flowers and costs \$50. A flat of geraniums contains 20 plants with red flowers and 10 plants with white flowers and costs \$40. A flat of petunias

contains 15 plants with red flowers and 5 plants with white flowers and costs \$35.

(a) [6pts.] What is the cheapest combination of flats of plants that landscaping can purchase to meet the campus's needs? [Hint: The simplex method is not the quickest way to analyze this.]

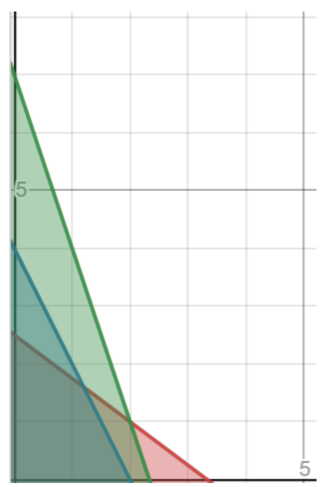
Solution: The problem above is to minimize $z = 50x_1 + 40x_2 + 35x_3$ subject to the constraints

$$\begin{cases} 15x_1 + 20x_2 + 15x_3 \geq 350 \\ 20x_1 + 10x_2 + 5x_3 \geq 300 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

with dual problem to maximize $z' = 350w_1 + 300w_2$ subject to the constraints

$$\begin{cases} 15w_1 + 20w_2 \leq 50 \\ 20w_1 + 10w_2 \leq 40 \\ 15w_1 + 5w_2 \leq 35 \\ w_1, w_2 \geq 0 \end{cases}$$

Since this is a two variable problem we can solve it via a graph fairly expeditiously. We have the following graph, where the triply shaded region is the region of feasible solutions.



The region of feasible solutions has corners $(0,0)$, $(2,0)$, $(0,5/2)$, $(6/5,8/5)$. Checking the values of the objective function at these corners we conclude that the maximum is $z' = 350(6/5)+300(8/5) = 420+480 = 900$. This must also be the optimal value of the original problem. To determine the optimal solution, we add some slack variables to the constraints, so that we have

$$\begin{cases} 15x_1 + 20x_2 + 15x_3 - u_1 = 350 \\ 20x_1 + 10x_2 + 5x_3 - u_2 = 300 \\ x_1, x_2, x_3, u_1, u_2 \geq 0 \end{cases}$$

and

$$\begin{cases} 15w_1 + 20w_2 + t_1 = 50 \\ 20w_1 + 10w_2 + t_2 = 40 \\ 15w_1 + 5w_2 + t_3 = 35 \\ w_1, w_2, t_1, t_2, t_3 \geq 0. \end{cases}$$

We see that in the optimal solution to the dual problem we have $t_3 > 0$, which by complementary slackness implies that in the optimal solution to the primal problem we have $x_3 = 0$. Similarly since in the optimal solution to the dual problem we have $w_1, w_2 > 0$, we conclude that in the optimal solution to the primal we have $u_1 = u_2 = 0$. So, we have the following equations at the optimal solution to the primal problem.

$$\begin{cases} 15x_1 + 20x_2 = 350 \\ 20x_1 + 10x_2 = 300 \\ x_1, x_2 \geq 0 \end{cases}$$

The solution to this is $x_1 = x_2 = 10$. So, Rutgers landscaping should buy 10 flats of poppies and 10 flats of geraniums, and no petunias, for a total cost of \$900.

(b) [4pts.] Because of campus expansion, the university expects that next year the number of plants with white flowers needed will be 500. What should the university's buying plan be next year? [Hint: Ditto.]

Solution: The objective function of the dual problem is now $z' = 350w_1 + 500w_2$ and it has the same region of feasible solutions. The optimal solution is now the corner $(0, 2.5)$ and the optimal value is 1250. Using the same names for the slack variables as above, we see that $t_2, t_3, w_2 > 0$ at the optimal solution to the dual problem, so $x_2, x_3, u_2 = 0$ in the optimal solution to the primal problem. We thus have the equations

$$\begin{cases} 15x_1 - u_1 = 350 \\ 20x_1 = 500 \\ x_1, x_2, x_3, u_1, u_2 \geq 0 \end{cases}$$

We conclude that $x_1 = 25$. So in this case landscaping should buy 25 flats of poppies (and no geraniums or petunias) for a total cost of \$1250.

3. [10pts.] An furniture company needs to ship office chairs from its four factories F_1, F_2, F_3, F_4 to its three distribution sites D_1, D_2, D_3. Below, the vector $\mathbf{s}$ gives the

supply at each factory, the vector **d** give this demand at each distribution site, and the matrix C gives the costs of shipping one unit between each pair of locations. Determine the cheapest shipping plan and its cost.

$$\mathbf{s} = \begin{bmatrix} 100 \\ 60 \\ 90 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} 60 \\ 80 \\ 70 \\ 40 \end{bmatrix}, \qquad C = \begin{bmatrix} 6 & 4 & 3 & 5 \\ 7 & 4 & 8 & 6 \\ 8 & 3 & 2 & 5 \end{bmatrix}.$$

Solution: We use the transportation algorithm. We start by setting up a transportation tableau.

$\boxed{6}$	$\boxed{4}$	$\boxed{3}$	$\boxed{5}$	100
$\boxed{7}$	$\boxed{4}$	$\boxed{8}$	$\boxed{6}$	60
$\boxed{8}$	$\boxed{3}$	$\boxed{2}$	$\boxed{5}$	90
60	80	70	40	

We will use Vogel's method to fill in an initial basic solution. We look at the difference between the cheapest route and second cheapest route in each row and column. The largest is the second row, at 2. So we populate that row.

$\boxed{6}$	$\boxed{4}$	$\boxed{3}$	$\boxed{5}$	100
$\boxed{7}$ 0	$\boxed{4}$ 60	$\boxed{8}$ 0	$\boxed{6}$ 0	60
$\boxed{8}$	$\boxed{3}$	$\boxed{2}$	$\boxed{5}$	90
60	80	70	40	

Ignoring filled-in entries, the new greatest difference between cheapest and second cheapest route is in the first column, so we populate it.

$\boxed{6}$ 60	$\boxed{4}$	$\boxed{3}$	$\boxed{5}$	100
$\boxed{7}$ 0	$\boxed{4}$ 60	$\boxed{8}$ 0	$\boxed{6}$ 0	60
$\boxed{8}$ 0	$\boxed{3}$	$\boxed{2}$	$\boxed{5}$	90
60	80	70	40	

Next we have a tie between several differences of 1. We pick the first row to populate. This is enough to specify the rest of the solution.

$\boxed{6}$ 60	$\boxed{4}$ 0	$\boxed{3}$ 40	$\boxed{5}$ 0	100
$\boxed{7}$ 0	$\boxed{4}$ 60	$\boxed{8}$ 0	$\boxed{6}$ 0	60
$\boxed{8}$ 0	$\boxed{3}$ 20	$\boxed{2}$ 30	$\boxed{5}$ 40	90
60	80	70	40	

We have six nonzero entries, so this is a nondegenerate basic solution. We now consider whether we can improve on it by solving the dual problem. The basic variables give us the following equations.

$$\begin{aligned} u_1 + v_1 &= 6 \\ u_1 + v_3 &= 3 \\ u_2 + v_2 &= 4 \\ u_3 + v_2 &= 3 \\ u_3 + v_3 &= 2 \\ u_3 + v_4 &= 5 \end{aligned}$$

which are solved by

$$u_1 = 1 \qquad u_2 = 1 \qquad u_3 = 0$$

$$v_1 = 5 \qquad v_2 = 3 \qquad v_3 = 2 \qquad v_4 = 5$$

We now consider the reduced costs associated with the nonbasic variables, and get the following.

$$
\begin{aligned}
x_{12} &: u_1 + v_2 - c_{12} = 1 + 3 - 4 = 0 \\
x_{14} &: u_1 + v_4 - c_{14} = 1 + 5 - 5 = 1 \\
x_{21} &: u_2 + v_1 - c_{21} = 1 + 5 - 7 = -1 \\
x_{23} &: u_2 + v_3 - c_{23} = 1 + 2 - 8 = -5 \\
x_{24} &: u_2 + v_4 - c_{24} = 1 + 5 - 6 = 0 \\
x_{31} &: u_3 + v_1 - c_{31} = 0 + 5 - 8 = -3
\end{aligned}
$$

We perceive that the entering variable is x_{14}, which gives us the pivot shown below.

$\boxed{6}$ 60	$\boxed{4}$ 0	$\boxed{3}$ - 40	$\boxed{5}$ + 0	100
$\boxed{7}$ 0	$\boxed{4}$ 60	$\boxed{8}$ 0	$\boxed{6}$ 0	60
$\boxed{8}$ 0	$\boxed{3}$ 20	$\boxed{2}$ + 30	$\boxed{5}$ - 40	90
60	80	70	40	

We see we can flow 40 through this loop, leaving us with the tableau below.

$\boxed{6}$ 60	$\boxed{4}$ 0	$\boxed{3}$ 0	$\boxed{5}$ 40	100
$\boxed{7}$ 0	$\boxed{4}$ 60	$\boxed{8}$ 0	$\boxed{6}$ 0	60
$\boxed{8}$ 0	$\boxed{3}$ 20	$\boxed{2}$ 70	$\boxed{5}$ 0	90
60	80	70	40	

We observe that we only have five nonzero entries, so we have a degenerate solution. We must choose one additional basic variable; we will let x_{34} continue to be basic. Above we have circled the basic variables so that we remember this. Now we solve

the dual problem to check optimality of the tableau. The basic variables give us the equations

$$\begin{aligned} u_1 + v_1 &= 6 \\ u_1 + v_4 &= 5 \\ u_2 + v_2 &= 4 \\ u_3 + v_2 &= 3 \\ u_3 + v_3 &= 2 \\ u_3 + v_4 &= 5 \end{aligned}$$

which are solved by

$$u_1 = 0 \qquad u_2 = 1 \qquad u_3 = 0$$
$$v_1 = 6 \qquad v_2 = 3 \qquad v_3 = 2 \qquad v_4 = 5.$$

We get the following reduced costs for the nonbasic variables.

$$\begin{aligned} x_{12} &: u_1 + v_2 - c_{12} = 0 + 3 - 4 = -1 \\ x_{13} &: u_1 + v_3 - c_{13} = 0 + 2 - 3 = -1 \\ x_{21} &: u_2 + v_1 - c_{21} = 1 + 6 - 7 = 0 \\ x_{23} &: u_2 + v_3 - c_{23} = 1 + 2 - 8 = -5 \\ x_{24} &: u_2 + v_4 - c_{24} = 1 + 5 - 6 = 0 \\ x_{31} &: u_3 + v_1 - c_{31} = 0 + 6 - 8 = -2 \end{aligned}$$

Since none of these numbers are positive, we conclude that the solution above is optimal, and that the minimum shipping cost is $C = 6(60) + 5(40) + 4(60) + 3(20) + 2(70) = 1000$.

Remark: In the problem on the final similar to this one, full credit will be given for any method of coming up with an initial solution – the advantage to you of knowing Vogel's method is that it on averages reduces the number of pivots you need to do subsequently.

4. (a) [5pts.] A courier service has five bicycle couriers $A_1, A_2, \ldots, A_5$ at various places in New Brunswick and five packages $P_1, \ldots P_5$ to be picked up. The number of minutes it will take the ith courier A_i to reach the pickup location for the package P_j is given as the entry c_{ij} of the matrix below. What assignment of couriers to packages minimizes the travel time?

$$\begin{bmatrix} 3 & 2 & 7 & 4 & 8 \\ 5 & 4 & 3 & 8 & 5 \\ 2 & 7 & 9 & 1 & 2 \\ 4 & 2 & 6 & 5 & 7 \\ 3 & 8 & 4 & 6 & 6 \end{bmatrix}$$

Solution: We can use the Hungarian method for solving assignment problems. First we subtract the smallest entry in each row from that row.

$$\begin{bmatrix} 1 & 0 & 5 & 2 & 6 \\ 2 & 1 & 0 & 5 & 2 \\ 1 & 6 & 8 & 0 & 1 \\ 2 & 0 & 4 & 3 & 5 \\ 0 & 5 & 1 & 3 & 3 \end{bmatrix}$$

Now we subtract the smallest entry in each column from that column, which only affects the final column.

$$\begin{bmatrix} 1 & 0 & 5 & 2 & 5 \\ 2 & 1 & 0 & 5 & 1 \\ 1 & 6 & 8 & 0 & 0 \\ 2 & 0 & 4 & 3 & 4 \\ 0 & 5 & 1 & 3 & 2 \end{bmatrix}$$

Now we try to make an assignment. We assign each row to the first 0 in an unoccupied column that we find in that row.

$$\begin{bmatrix} 1 & 0^* & 5 & 2 & 5 \\ 2 & 1 & 0^* & 5 & 1 \\ 1 & 6 & 8 & 0^* & 0 \\ 2 & 0 & 4 & 3 & 4 \\ 0^* & 5 & 1 & 3 & 2 \end{bmatrix}$$

We only succeed in assigning four rows. After checking potential paths from the 0 in the unassigned row we conclude that the second column is necessary. This makes the second, third, and fifth rows necessary. We cross them out.

$$\begin{bmatrix} 1 & 0^* & 5 & 2 & 5 \\ 2 & 1 & 0^* & 5 & 1 \\ 1 & 6 & 8 & 0^* & 0 \\ 2 & 0 & 4 & 3 & 4 \\ 0^* & 5 & 1 & 3 & 2 \end{bmatrix}$$

Now we subtract the smallest uncovered entry, which is 1, from all the uncovered entries and add it to the doubly-covered entries.

$$\begin{bmatrix} 0 & 0 & 4 & 1 & 4 \\ 2 & 2 & 0 & 5 & 1 \\ 1 & 7 & 8 & 0 & 0 \\ 1 & 0 & 3 & 2 & 3 \\ 0 & 6 & 1 & 3 & 2 \end{bmatrix}$$

We again attempt to make an assignment.

$$\begin{bmatrix} 0^* & 0 & 4 & 1 & 4 \\ 2 & 2 & 0^* & 5 & 1 \\ 1 & 7 & 8 & 0^* & 0 \\ 1 & 0^* & 3 & 2 & 3 \\ 0 & 6 & 1 & 3 & 2 \end{bmatrix}$$

We only succeed in assigning four rows. Checking paths from the 0 in the unassigned row leads us to conclude that the first two columns are necessary, which makes the second and third rows necessary. We cross them out.

$$\begin{bmatrix} 0^* & 0 & 4 & 1 & 4 \\ 2 & 2 & 0^* & 5 & 1 \\ 1 & 7 & 8 & 0^* & 0 \\ 1 & 0^* & 3 & 2 & 3 \\ 0 & 6 & 1 & 3 & 2 \end{bmatrix}$$

We subtract the smallest uncovered entry, which is 1, from each of the uncovered entries and add it to the doubly-covered entries, obtaining the following.

$$\begin{bmatrix} 0 & 0 & 3 & 0 & 3 \\ 3 & 3 & 0 & 5 & 1 \\ 2 & 8 & 8 & 0 & 0 \\ 1 & 0 & 2 & 1 & 2 \\ 0 & 6 & 0 & 2 & 1 \end{bmatrix}$$

We attempt to make an assignment.

$$\begin{bmatrix} 0^* & 0 & 3 & 0 & 3 \\ 3 & 3 & 0^* & 5 & 1 \\ 2 & 8 & 8 & 0^* & 0 \\ 1 & 0^* & 2 & 1 & 2 \\ 0 & 6 & 0 & 2 & 1 \end{bmatrix}$$

We again only succeed in assigning four rows, but this time there is an alternating path that fixes the problem.

$$\begin{bmatrix} 0^* & 0 & 3 & 0 & 3 \\ 3 & 3 & 0^* & 5 & 1 \\ 2 & 8 & 8 & 0^* & 0 \\ 1 & 0^* & 2 & 1 & 2 \\ 0 & 6 & 0 & 2 & 1 \end{bmatrix}$$

Now we can switch the 0's and 0^*'s in the alternating path. This gives us the following final assignment.

$$\begin{bmatrix} 0 & 0 & 3 & 0^* & 3 \\ 3 & 3 & 0^* & 5 & 1 \\ 2 & 8 & 8 & 0 & 0^* \\ 1 & 0^* & 2 & 1 & 2 \\ 0^* & 6 & 0 & 2 & 1 \end{bmatrix}$$

which on the original matrix is

$$\begin{bmatrix} 3 & 2 & 7 & 4^* & 8 \\ 5 & 4 & 3^* & 8 & 5 \\ 2 & 7 & 9 & 1 & 2^* \\ 4 & 2^* & 6 & 5 & 7 \\ 3^* & 8 & 4 & 6 & 6 \end{bmatrix}$$

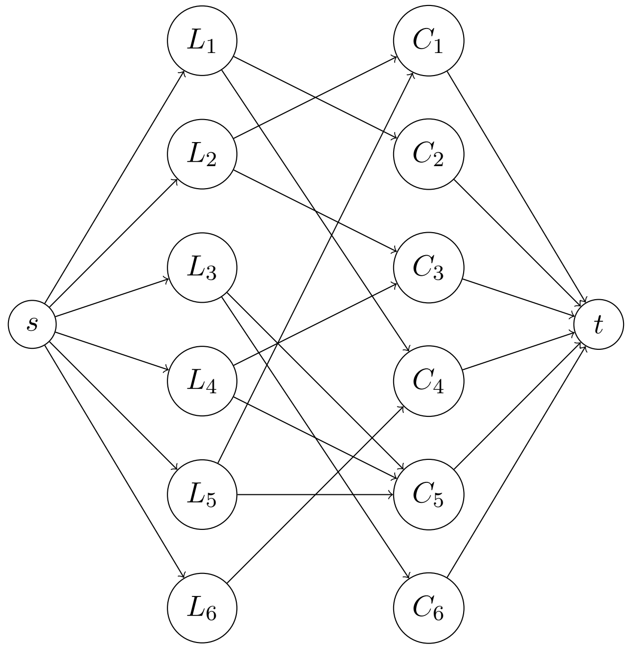
We conclude that the minimum total travel time is $4+3+2+2+3 = 14$ minutes.

(b) [5pts.] Six students $L_1, \ldots, L_6$ from a linear programming class plan to partner with six students $C_1, \ldots, C_6$ from a chemistry class to do final projects studying chemical reactions. Students would of course like to partner with their friends. The matrix of who is friends with whom is shown below; a 1 in the ij entry indicates that student L_i and C_j are friends and a 0 that they are not.

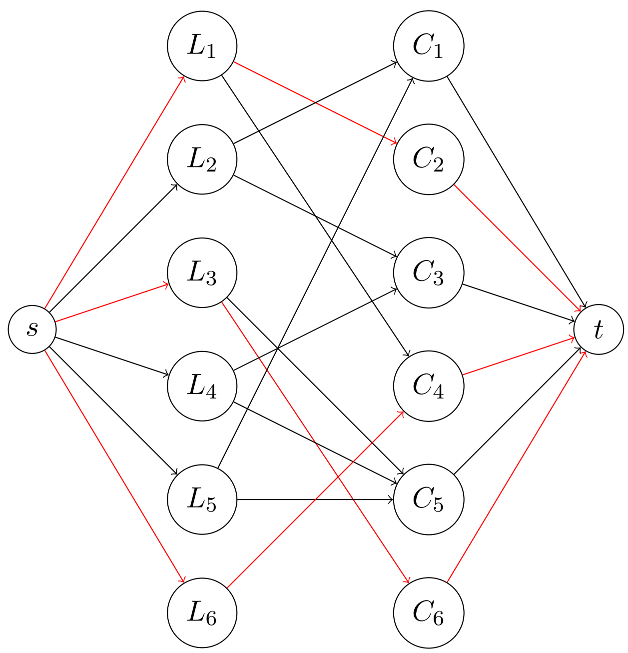
$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Determine the maximum number of partnerships that can be between two friends.

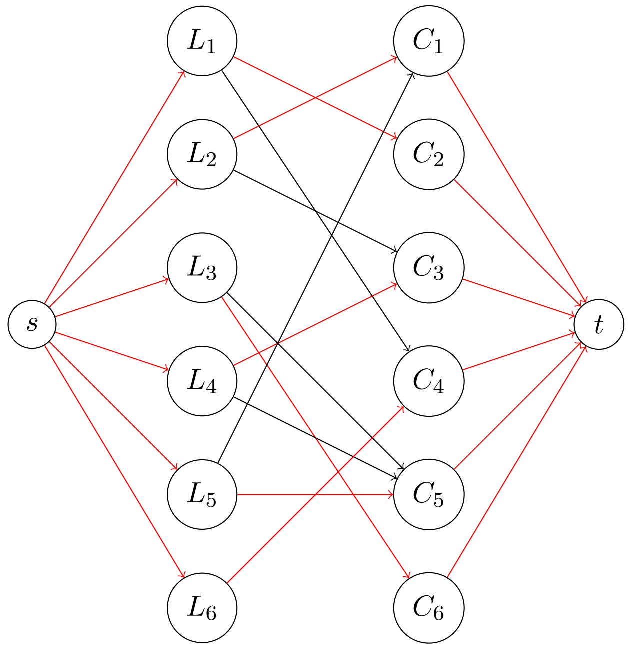
Solution: This can be done either as an assignment problem (being careful to maximize rather than minimize!) or a maximal flow problem; we present the maximal flow version. Our network looks like the following.



Since all capacities are 1, we'll just draw the ones that are currently being used in red below. We start by taking the three paths $s \to L_6 \to C_4 \to t$, $s \to L_3 \to C_6 \to t$, and $s \to L_1 \to C_2 \to t$.

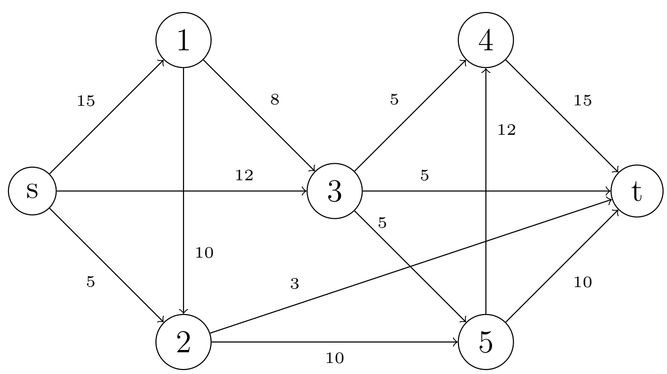


Now we add $s \to L_2 \to C_1 \to t$, $s \to L_4 \to C_3 \to t$, and $s \to L_5 \to C_5 \to t$.



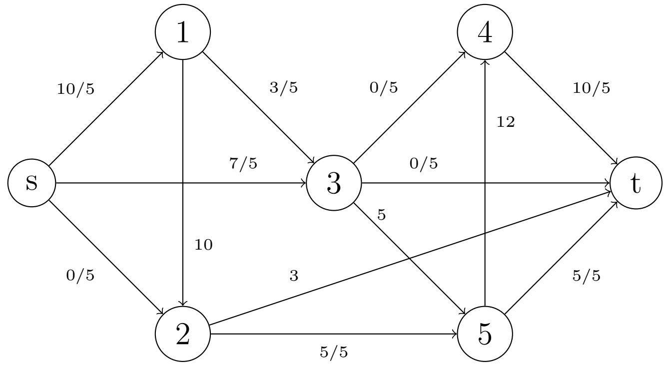
We are now at max flow of 6. It appears all six pairs can be between friends.

5. Consider the following pipeline network.

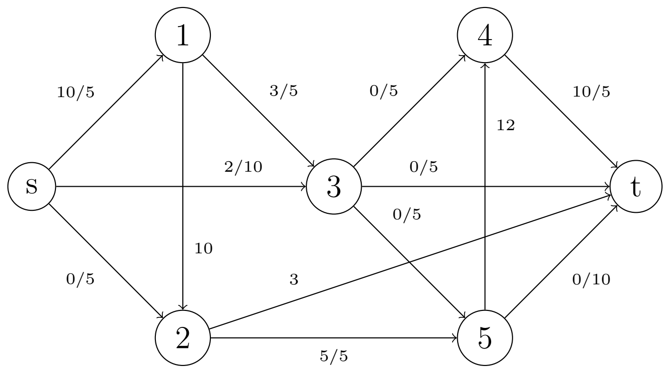


(a) [5pts.] Determine the maximum flow through this network.

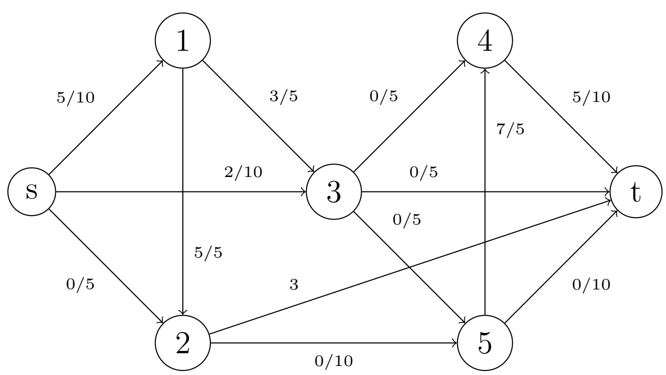
Solution: We proceed via the Ford-Fulkerson algorithm. We notice that each of the three paths $s \to 1 \to 3 \to 4 \to t$, $s \to 2 \to 5 \to t$, and $s \to 3 \to t$ has capacity 5, and they use nonoverlapping sets of edges, so we can start with those three paths, for a total flow of 15. Now we look at the residual network.



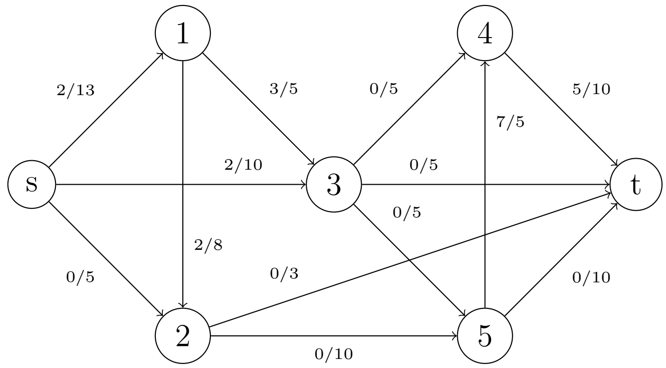
We see we can send an additional 5 along $s \to 3 \to 5 \to t$, bringing us to 20.



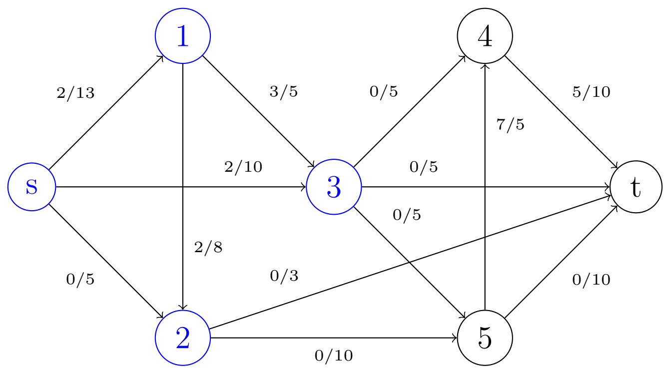
We can now send 5 through $s \to 1 \to 2 \to 5 \to 4 \to t$, taking us to 25.



We can now send 3 along $s \to 1 \to 2 \to t$, taking us to 28.

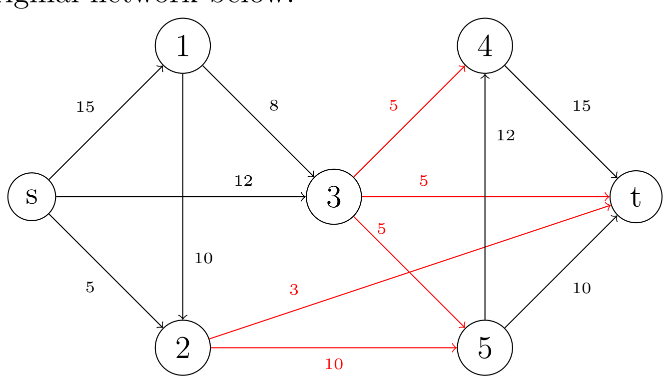


We are now done; the set of nodes reachable from the source by a positive capacity path in the residual network is shown below in blue.



(b) [3pts.] Locate a minimum cut in this network.

Solution: The set of directed edges from the blue nodes in the residual network shown above to the black nodes form a minimum cut; these edges are marked in red in the original network below.



(c) [2pts.] You can choose one pipeline to expand in order to increase the flow through the network. Which pipeline do you expand, and by how much can you usefully expand it?

Solution: We certainly can't change the maximum flow without altering something in the minimum cut we found above. It seems that by expanding either $3 \to 4$ or $3 \to t$ by 4 units we can get an additional 4 units of flow, along either the paths $s \to 1 \to 3 \to 4 \to t$ and $s \to 3 \to 4 \to t$ or the paths $s \to 1 \to 3 \to t$ and $s \to 3 \to t$. We clearly can't do any better, since this uses up all of the capacity outgoing from s.

6. Consider the following linear programming problem: maximize $z = 3x_1 + x_2 + 3x_3$ subject to

$$\begin{cases} x_1 - 3x_2 + 3x_3 \leq 30 \\ x_1 + x_2 + x_3 = 20 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

(a) [5pts.] Solve this problem using the simplex method.

Solution: We may add a slack variable and an artificial variable to the equations above to get the following.

$$\begin{cases} x_1 - 3x_2 + 3x_3 + u_1 = 30 \\ x_1 + x_2 + x_3 + y_1 = 20 \\ x_1, x_2, x_3, u_1, y_1 \geq 0 \end{cases}$$

For Phase I, we want to maximize $z = -y_1$. We start with the following.

	x_1	x_2	x_3	u_1	y_1	z	
u_1	1	−3	3	1	0	0	30
y_1	1	1	1	0	1	0	20
	0	0	0	0	1	1	0

We clear out the objective row to get an initial tableau.

	x_1	x_2	x_3	u_1	y_1	z	
u_1	1	−3	3	1	0	0	30
y_1	1	1	1	0	1	0	20
	−1	−1	−1	0	0	1	−20

We choose x_1 to be our entering variable, which makes y_1 the departing variable. We pivot.

	x_1	x_2	x_3	u_1	y_1	z	
u_1	0	−4	2	1	−1	0	10
x_1	1	1	1	0	1	0	20
	0	0	0	0	1	1	0

Phase I is now complete. We can drop the y_1 column and reintroduce the original objective function as follows.

	x_1	x_2	x_3	u_1	z	
u_1	0	-4	2	1	0	10
x_1	1	1	1	0	0	20
	-3	-1	-2	0	1	0

We clear out the objective row to get a tableau.

	x_1	x_2	x_3	u_1	z	
u_1	0	-4	2	1	0	10
x_1	1	1	1	0	0	20
	0	2	1	0	1	60

Since all the values in the objective row are nonnegative, we are now done. The optimal value is 60 and it is achieved at

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}.$$

(b) [2pts.] Suppose the objective function above is replaced with $z = 3x_1 + c_2' x_2 + 3x_3$. For what values of c_2' is the solution you found still optimal?

Solution: Replacing $c_2 = 1$ with $c_2' = 1 + \Delta c_2$ has the effect of changing the entry under x_2 in the objective row to $2 - \Delta c_2$. So, our solution is still optimal as long as $\Delta c_2 \leq 2$, or equivalently $c_2' \leq 3$.

(c) [3pts.] Suppose the first constraint above is replaced with $x_1 - 3x_2 + 3x_3 \leq 10$. What is the new optimal solution?

Solution: The change Δb_1 in the value of the constraint is -20. So, we add -20 times the column under the corresponding slack variable u_1 to the final column of our tableau, obtaining

	x_1	x_2	x_3	u_1	z	
u_1	0	-4	2	1	0	-10
x_1	1	1	1	0	0	20
	0	2	1	0	1	60

This is no longer feasible, so we must do a dual pivot back to feasibility. The

departing variable is u_1 and the entering variable is therefore x_2. We pivot.

	x_1	x_2	x_3	u_1	z	
x_2	0	1	$-1/2$	$-1/4$	0	$5/2$
x_1	1	0	$3/2$	$1/4$	0	$35/2$
	0	0	2	$1/2$	1	55

Now we are done. The new optimal value is 55 and the new optimal solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 35/2 \\ 5/2 \\ 0 \end{bmatrix}.$$