1. (a) \( -\frac{1}{\sqrt{n}} \leq \frac{\sin n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \). By the Squeeze Theorem, \( \lim_{n \to \infty} \frac{\sin n}{\sqrt{n}} = 0 \).

(b) Let \( f(x) = x \sin \left( \frac{1}{x} \right) \). Then
\[
\lim_{n \to \infty} a_n = \lim_{x \to \infty} x \sin \left( \frac{1}{x} \right) = \lim_{x \to \infty} \frac{\sin \left( \frac{1}{x} \right)}{\frac{1}{x}} = \lim_{t \to 0} \frac{\sin t}{t} = \lim_{t \to 0} \frac{\cos t}{1} = 1.
\]

(\text{Substitute } t = \frac{1}{x})

2. (20 points). Compute, using the method for surface area of a solid of revolution, the surface area of a sphere of radius \( R \).

\textbf{Solution} We can take the function \( f(x) = \sqrt{R^2 - x^2} \) on the domain \([-R, R]\). When rotated this gives a sphere of radius \( R \). The derivative of this function is \( \frac{x}{\sqrt{R^2 - x^2}} \). Let \( S \) denote the surface area. We have
\[
S = 2\pi \int_{-R}^{R} f(x) \sqrt{1 + (f'(x))^2} \, dx
= 2\pi \int_{-R}^{R} \sqrt{1 + \frac{x^2}{R^2 - x^2}} \sqrt{R^2 - x^2} \, dx
= 2\pi \int_{-R}^{R} \sqrt{R^2 - x^2} \, dx
= 2\pi \left[ x \sqrt{R^2 - x^2} + R^2 \arcsin \frac{x}{R} \right]_{-R}^{R}
= 4\pi R^2.
\]

3. (20 points). Compute the indefinite integral
\[
\int \frac{x + 7}{x^2(x + 2)} \, dx.
\]

\textbf{Solution} First, we write the integrand as a partial fraction
\[
\frac{x + 7}{x^2(x + 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 2}.
\]
where we determine $A, B, C$ now. Clear denominators to obtain

$$x + 7 = Ax(x + 2) + B(x + 2) + Cx^2 = (A + C)x^2 + (2A + B)x + 2B.$$ 

This yields the system of equations

$$A + C = 0$$
$$2A + B = 1$$
$$2B = 7.$$ 

Solving, we see that $B = \frac{7}{2}, A = -\frac{5}{4},$ and $C = \frac{5}{4}$. Thus, the integral is

$$\int \frac{x + 7}{x^2(x + 2)} \, dx = \int \left( -\frac{5}{4x} + \frac{7}{2x^2} + \frac{5}{4(x + 2)} \right) \, dx$$
$$= -\frac{5}{4} \int \frac{dx}{x} + \frac{7}{2} \int \frac{dx}{x^2} + \frac{5}{4} \int \frac{dx}{x + 2}$$
$$= -\frac{5}{4} \ln |x| - \frac{7}{2x} + \frac{5}{4} \ln |x + 2| + C.$$ 

4. (20 points) Find an interval $[a, b]$ containing $0$ such that if $x$ is in $[a, b]$, the error of the 5th Taylor polynomial for $f(x) = e^x$ (with $a = 0$) is less than or equal to $10^{-18}$.

**Solution** The error bound is

$$|T_5(x) - e^x| \leq \frac{K_6(x - 0)^6}{6!},$$

where $K_6$ is an upper bound for $f^6(x) = e^x$ on some interval as yet to be determined. So we solve for $x$ in the inequality

$$\frac{K_6 x^6}{6!} \leq 10^{-18}.$$ 

Let $b_0 = 1$, so that $e^{b_0} = e$. We have $b_0 > 0$ and on the interval $(-\infty, b_0)$, $e^x \leq e$. So, for any $x$ in the interval $(-\infty, b_0)$, the error is at most

$$\frac{ex^6}{6!} = \frac{ex^5}{360}.$$ 

Since $e \leq 3, \frac{e}{3} \leq 1$. Thus,

$$\frac{ex^6}{360} \leq \frac{x^6}{120}.$$ 

Set

$$x^6 \leq 120 \cdot 10^{-18}.$$ 

We can require even more strictly that

$$x^6 \leq 2^6 \cdot 10^{-18} = 64 \cdot 10^{-18}.$$
Then, we see that 
\[ |x| \leq 2 \cdot 10^{-3} \]
has the indicated error. Thus, on the interval \([-\frac{1}{500}, \frac{1}{500}]\), the error
\[ |T_n(x) - e^x| \]
is less than or equal to \(10^{-18}\).

5. (20 points). Compute the value of \(\ln 2\) to an error of at most \(10^{-3}\). You should use Taylor polynomials, but you do not have to actually simplify the final approximation \(T_n(2)\).

**Solution** We use the Taylor polynomials \(T_n(x)\) for \(f(x) = \ln x\) centered at \(a = 1\). The general form for the error bound is then,
\[ |T_n(2) - \ln 2| \leq \frac{K_{n+1}(2-1)^{n+1}}{(n+1)!}, \]
where \(K_{n+1}\) is an upper bound for \(f^{(n+1)}(x)\) on the interval \([1, 2]\). The first few derivatives of \(f(x)\) are
\[
\begin{align*}
    f'(x) & = \frac{1}{x} \\
    f''(x) & = -\frac{1}{x^2} \\
    f'''(x) & = \frac{2}{x^3} \\
    & \vdots 
\end{align*}
\]
The absolute values of these derivatives are all decreasing on the interval \([1, 2]\), so we can take their values at \(x = 1\) as upper bounds. In fact, we can take
\[ K_{n+1} = n! \]
for \(n \geq 0\). Now, we have that \(|f^{(n+1)}(x)| \leq K_{n+1}\) on the interval \([1, 2]\). Thus, we can take
\[ |T_n(2) - \ln 2| \leq \frac{n!}{(n+1)!} = \frac{1}{n+1} \]
Thus,
\[ |T_{999}(2) - \ln 2| \leq 10^{-3}. \]
The corresponding approximation is
\[ T_{999}(2) = \sum_{k=1}^{999} (-1)^{k-1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{1}{999}. \]