Math 311H
Honors Introduction to Real Analysis

Quiz

Instructions: You have 30 minutes to complete the quiz. There are three questions, worth a total of fifteen points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

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Question	Points	Score
1	5	
2	5	
3	5	
Total:	15	

1. For $n \in \mathbb{N}$, let P_n be the statement "$n^2 + 5n + 1$ is an even number".

   (a) [3pts.] Prove that if P_n is true, then P_{n+1} is true.

   Solution: Assume P_n is true, such that $n^2 + 5n + 1$ is even. Say it is equal to $2k$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned}(n+1)^2 + 5(n+1) + 1 &= (n^2 + 2n + 1) + 5n + 5 + 1 \\ &= (n^2 + 5n + 1) + 2n + 6 \\ &= 2k + 2n + 6 \\ &= 2(k + n + 3)\end{aligned}$$

   Ergo we see that $(n+1)^2 + 5(n+1) + 1$ is even, so P_{n+1} is true.

   (b) [1pts.] For which n is P_n true?

   Solution: None at all. For if n is even, then n^2 and $5n$ are also even, so $n^2 + 5n + 1$ is odd. But if n is odd, then n^2 and $5n$ are odd, so $n^2 + 5n$ is even, so $n^2 + 5n + 1$ is odd.

   (c) [1pts.] How do you reconcile parts (a) and (b)?

   Solution: At no point did we prove a base case for an inductive proof, only an inductive step.

2. (a) [4pts.] Let A and B be two nonempty bounded subsets of $\mathbb{R}$. Prove the equality $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

   Solution: Let $s = \sup A$ and $t = \sup B$. Up to relabeling we may assume $s \leq t$. We wish to show that $t = \sup(A \cup B)$. First, we claim t is an upper bound for $A \cup B$. For if $b \in B$, then by assumption t is an upper bound for B, so $b \leq t$, and if $a \in A$, then by assumption s is an upper bound for A, and $a < s \leq t$. So since any $x \in A \cup B$ has one of $x \in A$ or $x \in B$, we have $x \leq t$, so t is an upper bound for $A \cup B$. Now let u be any upper bound for $A \cup B$. Then since for all $b \in B$ we have $b \in A \cup B$, it follows that $b \leq u$, so u is also an upper bound for B. By assumption $t \leq u$. So, since t is an upper bound for $A \cup B$ and is less than or equal to any lower bound for $A \cup B$, we must have $t = \sup(A \cup B)$ as desired.

   (b) [1pts.] Give an example to show that it is not necessarily true that $\sup(A \cap B) = \min\{\sup A, \sup B\}$. Note that your example should have nonempty intersection.

Solution: Let $A = \{0, 1\}$ and $B = \{0, 2\}$, so that $\sup A = 1$ and $\sup B = 2$ but $\sup(A \cap B) = \sup(\{0\}) = 0$.

3. [5pts.] Recall that the product of two sets A and B is $A \times B = \{(a, b) : a \in A, b \in B\}$. Prove that the product of two countable sets is countable.

Solution: Let $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ be bijections. Let $f(n) = a_n$ and $g(n) = b_n$ so that $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$. We may construct a bijection $h : \mathbb{N} \to A \times B$ by making a grid of elements (a_i, b_j) and listing the elements along the diagonals, obtaining

$$(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_3, b_1), (a_2, b_2), (a_1, b_3), \dots$$

as the resulting list of elements, which is enough to define the map h.