Math 311H
Honors Introduction to Real Analysis

Midterm 2

Instructions: You have 80 minutes to complete the exam. There are six questions, worth a total of thirty points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

Question	Points	Score
1	5	
2	5	
3	4	
4	6	
5	5	
6	5	
Total:	30	

1. For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.

   (a) [1pts.] A nonempty perfect set consisting only of rational numbers.

   Solution: Impossible; perfect sets are always uncountable.

   (b) [1pts.] A connected set consisting of only nonzero numbers.

   Solution: Consider $(1, 2)$.

   (c) [1pts.] A function $f \colon \mathbb{R} \to \mathbb{R}$ which is continuous at 0 and no other point.

   Solution: Consider

$$h(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

   (d) [1pts.] A continuous function $f \colon [a, b] \to \mathbb{R}$ and a Cauchy sequence (s_n) of elements in $[a, b]$ such that $(f(s_n))$ is not Cauchy.

   Solution: Impossible; continuous functions on compact sets are uniformly continuous, and the image of a Cauchy sequence under a uniformly continuous function is Cauchy.

   (e) [1pts.] A closed set E for which $E^\circ \neq 0$ and $\overline{(E^\circ)} \neq E$.

   Solution: Consider $E = [0, 1] \cup \{2\}$, so that $E^\circ = (0, 1)$ and $\overline{(E^\circ)} = [0, 1]$.

2. (a) [3pts.] Let $g \colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that if $g(r) = 0$ for all points $r \in \mathbb{Q}$, then in fact $g(x) = 0$ for all $x \in \mathbb{R}$.

   Solution: For any $x \in \mathbb{R}$, pick a sequence of rational numbers $r_n \to x$. Then by continuity of g, we have $g(r_n) \to g(x)$. But $g(r_n) = 0$ for all n, implying that in fact $g(x) = 0$.

   (b) [2pts.] Prove that it follows that if $f, h \colon \mathbb{R} \to \mathbb{R}$ are continuous and $f(r) = h(r)$ for all rational numbers, then $f(x) = h(x)$ for all $x \in \mathbb{R}$.

   Solution: We may set $g = f - h$ and apply part (a).

3. [4pts.] (The Squeeze Theorem for Limits of Functions) Let f, g, h be functions with the same domain A and let $f(x) \leq g(x) \leq h(x)$ for all $x \in A$. If $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$ for c some limit point of A, prove that $\lim_{x \to c} g(x) = L$ as well.

Solution: Let (x_n) be a sequence of points in A converging to c such that $x_n \neq c$ for any n. Since $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, we have that $f(x_n) \to L$ and $h(x_n) \to L$. However, since $f(x_n) \leq g(x_n) \leq h(x_n)$, we see that by the Squeeze Theorem for sequences, the sequence $(g(x_n))$ also converges to L. As (x_n) was arbitrary, $\lim_{x \to c} g(x) = L$.

4. For each of the following pairs of sets, either give an example of a continuous function $f \colon A \to \mathbb{R}$ whose image is $f(A) = B$ (no need to justify your answer) or explain why no such function exists.

(a) [2pts.] $A = (0, \infty)$; $B = [1, 2]$.

Solution: This is possible, let

$$f(x) = \begin{cases} 1 & x < 1 \\ x & 1 \leq x \leq 2 \\ 2 & x > 2 \end{cases}$$

(b) [2pts.] $A = (0, 1) \cup (2, 3)$; $B = (0, 1) \cup (2, 3) \cup (4, 5]$.

Solution: Impossible. By the Intermediate Value Theorem, the image of an interval under a continuous function is an interval, implying that each of $f((0, 1))$ and $f((2, 3))$ are contained in one of the three intervals in B. This leaves an interval in B outside the image of f.

(c) [2pts.] A is the Cantor set; $B = [0, 1) \cup (2, 3]$.

Solution: Impossible; A is compact and B is not.

5. Let $f \colon \mathbb{R} \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$.

(a) [2pts.] Prove or disprove: $h(x) = f(x) + g(x)$ is uniformly continuous on $\mathbb{R}$.

Solution: True. Let $\epsilon > 0$, then there exists $\delta_1 > 0$ such that $|x - y| < \delta_1$ implies that $|f(x) - f(y)| < \frac{\epsilon}{2}$, and there exists $\delta_2 > 0$ such that $|x - y| < \delta_2$

implies that $|g(x) - g(y)| < \frac{\epsilon}{2}$. Ergo if $\delta = \min\{\delta_1, \delta_2\}$, we have that $|x - y| < \delta$ implies that

$$\begin{aligned} |h(x) - h(y)| &= |f(x) + g(x) - (f(y) + g(y))| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

As $\epsilon > 0$ was arbitary, $h(x)$ is uniformly continuous on $\mathbb{R}$.

(b) [3pts.] Prove or disprove: $k(x) = f(x)g(x)$ is uniformly continuous on $\mathbb{R}$.

Solution: False; the function $f(x) = x$ is trivially uniformly continuous on $\mathbb{R}$, but from class $[f(x)]^2 = x^2$ is not.

6. A continuous map $f\colon \mathbb{R} \to \mathbb{R}$ is called *proper* if the preimage $f^{-1}(K) = K$ of every compact set is a compact set.

(a) [1pts.] Give an example of a map $f\colon \mathbb{R} \to \mathbb{R}$ which is not proper.

Solution: Consider the constant function $f(x) \equiv 0$, such that $f^{-1}(0) = \mathbb{R}$.

(b) [2pts.] Prove that the image of a proper map $f\colon \mathbb{R} \to \mathbb{R}$ is necessarily unbounded.

Solution: Suppose not, that is, suppose that $f(\mathbb{R})$ is bounded. Then $f(\mathbb{R})$ is contained in some closed interval K. We see that $f^{-1}(K) = \mathbb{R}$, but f was proper, so we have a contradiction. We conclude $f(\mathbb{R})$ is unbounded.

(c) [2pts.] Give an example of a map $f\colon \mathbb{R} \to \mathbb{R}$ which is proper and whose image is not all of $\mathbb{R}$.

Solution: Consider $f(x) = |x|$, which has image the nonnegative real numbers. Let K be compact. Then K is bounded, which is to say that there exists M such that for $x \in K$, we have $|x| < M$. This clearly implies that $f^{-1}(K) = \{x : |x| \in K\}$ is also bounded by M. Moreover, K is closed. By continuity of f, we see that $f^{-1}(K)$ is also closed. Ergo, $f^{-1}(K)$ is compact. As K was arbitrary, f is proper.