Math 311H
Honors Introduction to Real Analysis

Sample Midterm 2

Instructions: You have 80 minutes to complete the exam. There are six questions, worth a total of thirty points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ____

Question	Points	Score
1	5	
2	5	
3	4	
4	5	
5	6	
6	5	
Total:	30	

1. For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.

   (a) [1pts.] A perfect set with exactly three limit points.

   Solution: Impossible; every point of a perfect set is a limit point and perfect sets are always uncountable.

   (b) [1pts.] A set E with $E^\circ = \emptyset$ and $\overline{E} = \mathbb{R}$.

   Solution: Consider $\mathbb{Q}$.

   (c) [1pts.] A connected set consisting of only irrational numbers.

   Solution: Consider $\{\sqrt{2}\}$.

   (d) [1pts.] A noncompact set A and an open cover of A which has a finite subcover.

   Solution: Let $A = (0, 1)$ and consider the open cover $\{A\}$, which is already finite.

   (e) [1pts.] A continuous surjective function from the Cantor set to the interval $[0, 1]$.

   Solution: Recall that the Cantor set consists of all numbers in $[0, 1]$ with decimal expansions in base three containing only 0 and 2 as digits. The map is division by two and reinterpretation as an expansion in base two (so that $\frac{1}{3} \mapsto \frac{1}{2}$ and so on).

2. [5pts.] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $E \subseteq R$. Prove that $f(\overline{E}) \subseteq \overline{f(E)}$.

   Solution: Since the closure of any set is closed and the preimage of a closed set under a continuous function is closed, we have that $f^{-1}(\overline{f(E)})$ is closed. Moreover, since if $x \in E$, we have $f(x) \subset f(E) \subset \overline{f(E)}$, so $x \in f^{-1}(\overline{f(E)})$. Hence $E \subset f^{-1}(\overline{f(E)})$. But $\overline{E}$ is the smallest closed set containing E, so this implies that $\overline{E} \subset f^{-1}(\overline{f(E)})$. Ergo $f(\overline{E}) \subseteq \overline{f(E)}$. A direct argument involving checking that any limit point of E is sent to either a point of $f(E)$ or a limit point of $f(E)$ is also possible.

3. [4pts.] Suppose that f, g are two functions with the same domain A such that $f(x) \leq g(x)$ for all $x \in A$, and say that $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} L_2$ both exist for some limit point c of A. Prove that $L_1 \leq L_2$.

Solution: Let (x_n) be a sequence of points in A such that $x_n \to c$ but $x_n \neq c$ for any n. Then $f(x_n) \to L_1$ and $g(x_n) \to L_2$ by the sequential criterion for limits. However, $f(x_n) \leq g(x_n)$ for all n, so by the Order Limit Theorem for sequences, we must have that $L_1 \leq L_2$.

4. [5pts.] Let f be uniformly continuous on a bounded set A. Prove that the image $f(A)$ is also bounded.

Solution: Suppose not, then there is a sequence of points (y_n) in $f(A)$ such that $|y_n| > n$ for all $n \in \mathbb{N}$. Now, since $y_n \in f(A)$, there is some $x_n \in A$ such that $f(x_n) = y_n$. The sequence (x_n) is bounded, hence by Bolzano-Weierstrass it has a subsequence (x_{n_k}) which converges in $\mathbb{R}$ and in particular is Cauchy. But since f is uniformly continuous, the sequence $(f(x_{n_k})) = (y_{n_k})$ should be Cauchy as well, and in particular bounded. This is a contradiction since $y_{n_k} > n_k$ for all n_k.

5. For each of the following pairs of sets, either give an example of a continuous function $f: A \to \mathbb{R}$ whose image is $f(A) = B$ (no need to justify your answer) or explain why no such function exists.

   (a) [2pts.] $A = [0, 1]$; $B = [1, 2)$

   Solution: Impossible; A is compact and B is not.

   (b) [2pts.] $A = (0, 1]$; $B = [1, \infty)$

   Solution: Possible; consider $f(x) = \frac{1}{x}$.

   (c) [2pts.] $A = (0, 1)$; $B = (0, 1) \cup (3, 4)$

   Solution: Impossible; A is connected and B is not.

6. A map $f: \mathbb{R} \to \mathbb{R}$ is called *open* if for every $O \subset \mathbb{R}$ an open set, the image $f(O)$ is also open.

   (a) [1pts.] Give an example of a continuous function $f: \mathbb{R} \to \mathbb{R}$ which is not open, including an open set O such that $f(O)$ is not open.

   Solution: Consider $f(x) = |x|$, which has the property that $f((-1, 1)) = [0, 1)$.

   (b) [4pts.] Prove that any open continuous map f is strictly monotone. [Hint: For any $a < b$ in $\mathbb{R}$, where must the minimum and maximum values of f on $[a, b]$ lie?]

Solution: Let $a \in \mathbb{R}$, and consider any b such that $a < b$. We see that $f([a, b])$ is a compact connected set, hence either a closed interval or a point; since $f((a, b))$ is open, we in fact have that $f([a, b])$ is a closed interval $[c, d]$ and that c and d are not in the image $f((a, b))$, so they must be the image of a and b in some order. In particular, the maximum and minimum values of f on a closed interval occur at the endpoints.

We now have two cases. Suppose that $f(a) = c$ and $f(b) = d$, so that $f(a) < f(b)$. In this case, if $a < x < b$, then $f(x) \in (c, d)$, which implies that $f(a) < f(x)$, and if $a < b < y$, then the same argument shows that a and y map to the endpoints of an interval which contains $f(b)$, so a must also be the left-hand endpoint of that interval. Hence $f(a) < f(y)$. So we see that $f(a) < f(x)$ for all $x > a$. If we instead assumed that $f(a) = d$ and $f(b) = c$, we would have gotten $f(a) > f(x)$ for all $x > a$. So one of these things is true for every $a \in \mathbb{R}$. We will now show that it must be the same condition for every a.

Now, suppose we have an a in $\mathbb{R}$ with the property that $f(a) < f(x)$ for all $x > a$ and an a' in $\mathbb{R}$ with the property that $f(a') > f(x)$ for all $x > a'$. Then suppose $a < a'$, and $x > a'$. Then we should have $f(a) < f(a')$ and $f(a) < f(x)$, but also we should have $f(a') > f(x)$. So, in particular, $f([a, x])$ does not take its maximum at either a or x, contradicting openness. Hence, we only get points of one type, so f is strictly monotone.

Remark: The corresponding problem on the actual exam is easier than this one.