Math 311H
Honors Introduction to Real Analysis
Midterm 2

Instructions: You have 80 minutes to complete the exam. There are six questions, worth a total of thirty points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

Question	Points	Score
1	5	
2	5	
3	4	
4	6	
5	5	
6	5	
Total:	30	

1. For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.

   (a) [1pts.] A nonempty perfect set consisting only of rational numbers.

   (b) [1pts.] A connected set consisting of only nonzero numbers.

   (c) [1pts.] A function $f\colon \mathbb{R} \to \mathbb{R}$ which is continuous at 0 and no other point.

   (d) [1pts.] A continuous function $f\colon [a, b] \to \mathbb{R}$ and a Cauchy sequence (s_n) of elements in $[a, b]$ such that $(f(s_n))$ is not Cauchy.

   (e) [1pts.] A closed set E for which $E^\circ \neq 0$ and $\overline{(E^\circ)} \neq E$.

2. (a) [3pts.] Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that if $g(r) = 0$ for all points $r \in \mathbb{Q}$, then in fact $g(x) = 0$ for all $x \in \mathbb{R}$.

   (b) [2pts.] Prove that it follows that if $f, h\colon \mathbb{R} \to \mathbb{R}$ are continuous and $f(r) = h(r)$ for all rational numbers, then $f(x) = h(x)$ for all $x \in \mathbb{R}$.

3. [4pts.] (The Squeeze Theorem for Limits of Functions) Let f, g, h be functions with the same domain A and let $f(x) \leq g(x) \leq h(x)$ for all $x \in A$. If $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$ for c some limit point of A, prove that $\lim_{x\to c} g(x) = L$ as well.

4. For each of the following pairs of sets, either give an example of a continuous function $f\colon A \to \mathbb{R}$ whose image is $f(A) = B$ (no need to justify your answer) or explain why no such function exists.

   (a) [2pts.] $A = (0, \infty)$; $B = [1, 2]$.

   (b) [2pts.] $A = (0, 1) \cup (2, 3)$; $B = (0, 1) \cup (2, 3) \cup (4, 5]$.

   (c) [2pts.] A is the Cantor set; $B = [0, 1) \cup (2, 3]$.

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$.

   (a) [2pts.] Prove or disprove: $h(x) = f(x) + g(x)$ is uniformly continuous on $\mathbb{R}$.

   (b) [3pts.] Prove or disprove: $k(x) = f(x)g(x)$ is uniformly continuous on $\mathbb{R}$.

6. A continuous map $f\colon \mathbb{R} \to \mathbb{R}$ is called *proper* if the preimage $f^{-1}(K) = K$ of every compact set is a compact set.
   (a) [1pts.] Give an example of a map $f\colon \mathbb{R} \to \mathbb{R}$ which is not proper.
   (b) [2pts.] Prove that the image of a proper map $f\colon \mathbb{R} \to \mathbb{R}$ is necessarily unbounded.
   (c) [2pts.] Give an example of a map $f\colon \mathbb{R} \to \mathbb{R}$ which is proper and whose image is not all of $\mathbb{R}$.