Math 311H
Honors Introduction to Real Analysis

Midterm 1

Instructions: You have 80 minutes to complete the exam. There are six questions, worth a total of thirty points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

Question	Points	Score
1	6	
2	4	
3	5	
4	5	
5	5	
6	5	
Total:	30	

1. For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.

   (a) [1pts.] A Cauchy sequence with no monotone subsequence.

   Solution: This is impossible; every sequence has a monotone subsequence.

   (b) [1pts.] A monotone sequence with no Cauchy subsequence.

   Solution: Consider $(1, 2, 3, 4, \dots)$.

   (c) [1pts.] A sequence with exactly three subsequential limits.

   Solution: Consider $(1, 2, 3, 1, 2, 3, 1, 2, 3, \dots)$.

   (d) [1pts.] A series $\sum_{n=1}^{\infty} a_n$ for which $\sum_{n=1}^{\infty} |a_n|$ converges but $\sum_{n=1}^{\infty} a_n$ does not.

   Solution: This is impossible; absolutely convergent series converge.

   (e) [1pts.] An alternating series $\sum_{n=1}^{\infty} a_n$ of rational numbers converging to $\sqrt{2}$, and a partial sum of this series which is within .01 of $\sqrt{2}$.

   Solution: Consider the series $2 - .6 + .02 - .006 + \dots$ oscillating around the decimal expansion of the square root of 2. Since the error on the partial sums of an alternating series is always less than the absolute value of the next term of the series, we see that based on the fourth partial sum of the series being 1.414 and having error of less than .001, the sum of this series is between 1.414 and 1.415 and in particular within .01 of $\sqrt{2}$.

   (f) [1pts.] A sequence of nonempty closed sets $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ such that $\bigcap_{n=1}^{\infty} F_n$ is empty.

   Solution: Consider $F_n = [n, \infty)$, so that $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

2. (a) [3pts.] Let (a_n) be a sequence with the property that $|a_n - a_{n+1}| < \frac{1}{2^n}$. Show that (a_n) is Cauchy.

   Solution: Suppose that (a_n) is a sequence such that $|a_n - a_{n+1}| < \frac{1}{2^n}$. Given $\epsilon > 0$, choose N such that $\frac{1}{2^{N-1}} < \epsilon$. Then if we have $n > m \geq N$, we compute that

$$\begin{aligned}
|a_m - a_n| &= |(a_m - a_{m+1}) + (a_{m+1} - a_{m+2}) + \cdots + (a_{n-1} - a_n)| \\
&\leq |(a_m - a_{m+1})| + |(a_{m+1} - a_{m+2})| + \cdots + |(a_{n-1} - a_n)| \\
&< \frac{1}{2^m} + \frac{1}{2^{m-1}} + \cdots + \frac{1}{2^{n-1}} \\
&< \sum_{k=0}^{\infty} \frac{1}{2^m} \cdot \frac{1}{2^k} \\
&= \frac{1}{2^{m-1}} \\
&\leq \frac{1}{2^{N-1}} \\
&< \epsilon
\end{aligned}$$

Since ϵ was arbitary we conclude the sequence is Cauchy.

(b) [1pts.] Is the conclusion of part (a) true if the condition is instead $|a_n - a_{n+1}| < \frac{1}{n}$?

Solution: No. For example the condition is true of the sequence $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, which does not converge and hence is not Cauchy.

3. Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \frac{5a_n}{3+a_n}$.

(a) [3pts.] Prove that $1 \leq a_n \leq 2$ for all n and (a_n) is increasing.

Solution: We start by showing the bound. Since $a_1 = 1$, the base case $1 \leq a_0 \leq 2$ is clearly true. Now suppose that $1 \leq a_n \leq 2$. We observe that

$$a_{n+1} \geq 1 \Leftrightarrow \frac{5a_n}{3+a_n} \geq 1 \Leftrightarrow 5a_n \geq 3 + a_n \Leftrightarrow 4a_n \geq 3 \Leftrightarrow a_n \geq \frac{3}{4}$$

where in the second step we use the fact that $3 + a_n$ is positive, so since $a_n \geq 1 > \frac{3}{4}$, we conclude that $a_{n+1} \geq 2$. Likewise

$$a_{n+1} \leq 2 \Leftrightarrow \frac{5a_n}{3+a_n} \leq 2 \Leftrightarrow 5a_n \leq 6 + 2a_n \Leftrightarrow 3a_n \leq 6 \Leftrightarrow a_n \leq 2$$

so $a_{n+1} \leq 2$ is also true.

Now we check that the sequence is increasing, that is that $a_{n+1} - a_n \geq 0$. Observe that

$$a_{n+1} - a_n = \frac{5a_n}{3+a_n} - a_n = \frac{5a_n - 3a_n - a_n^2}{3+a_n} = \frac{(2-a_n)(a_n)}{3+a_n}$$

Since $1 \le a_n \le 2$, this expression is clearly nonnegative. So $a_{n+1} \ge a_n$ for all n.

(b) [2pts.] Prove that (a_n) converges and compute the limit, justifying your steps carefully.

Solution: From part (a), the sequence (a_n) is bounded monotone, hence convergent. Applying the algebraic limit theorems to the relationship $a_n = \frac{5a_n}{3+a_n}$ we obtain $a = \frac{5a}{3+a}$, or $3a + a^2 = 5a$, which simplifies to $a(a-2) = 0$. By the order limit theorem, the only solution to this which can be the limit of the sequence is 2.

4. Let $\sum_{n=1}^{\infty} a_n$ be a series with the property that $\lim a_n = 0$.

(a) [1pts.] Give an example to show that $\sum_{n=1}^{\infty} a_n$ need not necessarily converge.

Solution: Consider $\sum_{n=1}^{\infty} \frac{1}{n}$.

(b) [4pts.] Prove that there exists a subsequence (a_{n_k}) of (a_n) with the property that $\sum_{k=1}^{\infty} a_{n_k}$ converges. [Hint: Start by arguing that there is a subsequence (a_{n_k}) with the property that $|a_{n_k}| \le \frac{1}{k^2}$.]

Solution: Choose a subsequence (a_{n_k}) as follows. Since $a_n \to 0$, there is some a_{n_1} such that $|a_{n_1}| < 1$. Having chosen this n_1, choose n_2 such that $n_1 < n_2$ and $|a_{n_2}| < \frac{1}{4}$, which must exist since otherwise a_n does not converge to zero. Inductively, continue picking a subsequence such that $n_1 < n_2 < n_3 < \dots$ and $|a_{n_k}| < \frac{1}{k^2}$. Then by the Comparison Test, the series $\sum_{k=1}^{\infty} a_{n_k}$ converges.

5. Let A_n be a subset of $\mathbb{R}$ for every n.

(a) [3pts.] Prove that $\cup_{n=1}^{\infty} A_n^\circ \subseteq (\cup_{n=1}^{\infty} A_n)^\circ$.

Solution: Suppose that $x \in \bigcup_{n=1}^{\infty} A_n^\circ$. Then in particular $x \in A_n^\circ$ for some n, implying that there is some $V_\epsilon(x) \subset A_n$. But then $V_\epsilon(x) \subset \bigcup_{n=1}^{\infty} A_n$, so we see that $x \in (\bigcup_{n=1}^{\infty} A_n)^\circ$. The inclusion follows.

(You can also do this by noting that $\bigcup_{n=1}^{\infty} A_n^\circ$ is an open set contained in $\bigcup_{n=1}^{\infty} A_n$ and therefore is a subset of $(\bigcup_{n=1}^{\infty} A_n)^\circ$.)

(b) [2pts.] Give an example to show the inclusion above may be proper.

Solution: Consider, eg, $A_1 = \mathbb{R} \setminus \mathbb{Q}$, and letting the remaining sets $A_n = \{r_{n-1}\}$ each contain a single rational number in some enumeration of the rationals

$(r_1, r_2, r_3, \dots)$, so that each set has empty interior but the interior of their union is all of $\mathbb{R}$.

6. A sequence (a_n) is said to *diverge to infinity* if, for every $M > 0$, there exists N such that $n \geq N$ implies that $a_n > M$.

   (a) [1pts.] Give an example (no need to justify it) of a sequence that diverges to infinity.

   Solution: THe sequence $(1, 2, 3, 4, \cdots)$ diverges to infinity.

   (b) [1pts.] Give an example (no need to justify it) of a sequence that diverges but does not diverge to infinity.

   Solution: The sequence $(1, -1, 1, -1, \dots)$ is an example.

   (c) [2pts.] Prove that if (a_n) is a sequence that diverges to infinity and (b_n) converges to a limit $b > 0$, then $(a_n b_n)$ also diverges to infinity.

   Solution: Let $M > 0$. Choose N_1 such that $n \geq N_1$ implies that $a_n > \frac{2M}{b}$. Furthermore, choose N_2 such that $n \geq N_2$ implies that $|b - b_n| < \frac{b}{2}$, so that in particular $n \geq N_2$ implies that $b_n > \frac{b}{2}$. Then $n \geq N = \max\{N_1, N_2\}$ implies that $a_n b_n > \left(\frac{2M}{b}\right)\left(\frac{b}{2}\right) = M$. As M was arbitrary, we are done.

   (d) [1pts.] Give an example to show that the statement above is not true if the condition $b > 0$ is removed.

   Solution: Let (a_n) such that $a_n = n$ be our sequence diverging to infinity. If $b_n = (-1)^{n+1}$, then $(a_n b_n) = (1, -2, 3, -4, \dots)$ diverges but not to infinity. For any real number r, if $b_n = \frac{r}{n}$, then $(a_n b_n) = (r)$ converges to r. If $b_n = -1$, then $(a_n b_n)$ diverges to $-\infty$. Any of these examples shows the claim above does not hold if we remove $b > 0$.