Math 311H
Honors Introduction to Real Analysis

Sample Midterm 1

Instructions: You have 80 minutes to complete the exam. There are six questions, worth a total of thirty points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

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Question	Points	Score
1	6	
2	4	
3	5	
4	5	
5	5	
6	5	
Total:	30	

1. For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.

   (a) [1pts.] A sequence which is not monotone and has no convergent subsequence.

   Solution: Consider the sequence $(1, -1, 2, -2, 3, -3, \dots)$.

   (b) [1pts.] A Cauchy sequence with a divergent subsequence.

   Solution: This is impossible. Cauchy sequences are convergent, and every subsequence of a convergent sequence converges.

   (c) [1pts.] A sequence with exactly two subsequential limits.

   Solution: Consider $(1, -1, 1, -1, \dots)$.

   (d) [1pts.] A bounded sequence with no Cauchy subsequence.

   Solution: This is impossible. By Bolzano-Weierstrass, a bounded sequence has a subsequence which converges, hence is Cauchy.

   (e) [1pts.] A bounded above nonempty subset A of $\mathbb{R}$ which contains its supremum but is not closed.

   Solution: Consider $A = (1, 2]$.

   (f) [1pts.] A series whose sum is 3.

   Solution: Consider $\sum_{n=1}^{\infty} \frac{3}{2^n}$.

2. [4pts.] Suppose that a_n and b_n are Cauchy sequences. Prove directly that $(a_n b_n)$ is a Cauchy sequence. ["Directly" here means that your proof should not reference the fact that Cauchy sequences converge in $\mathbb{R}$.]

   Solution: Cauchy sequences are bounded, so we may choose M_1 such that $|a_n| < M_1$ for all n and $|b_n| < M_2$ for all n. Let $M = \max\{M_1, M_2\}$. Given $\epsilon > 0$, pick N_1 such that $n, m \geq N_1$ implies that $|a_n - a_m| < \frac{\epsilon}{2M}$ and N_2 such that $n, m \geq N_2$ implies that $|b_n - b_m| < \frac{\epsilon}{2M}$. Then $n, m \geq N = \max\{N_1, N_2\}$ implies that

$$\begin{aligned}|a_nb_n - a_mb_m| &= |a_nb_n - a_nb_m + a_nb_m - a_mb_m| \\ &\leq |a_nb_n - a_nb_m| + |a_nb_m - a_mb_m| \\ &= |a_n||b_n - b_m| + |a_n - a_m||b_m| \\ &< M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2M} \cdot M \\ &= \epsilon.\end{aligned}$$

3. [5pts.] Prove the *Root Test*: Let $\sum_{n=1}^{\infty} a_n$ be a series with the property that $\lim |a_n|^{\frac{1}{n}}$ exists and is equal to $L < 1$. Prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Solution: We may compare to a geometric series, as follows. Choose k such that $L < k < 1$. Then there is some N such that $n \geq N$ implies that $|a_n|^{\frac{1}{n}} < k$, or in other words $|a_n| < k^n$ for $n \geq N$. By the Comparison Test, $\sum_{n+N}^{\infty} a_n$ converges absolutely. However, this implies that $\sum_{n=1}^{\infty} a_n$ converges absolutely. So we are done.

4. [5pts.] Consider the sequence defined recursively by $a_0 = 1$ and $a_{n+1} = 2(a_n)^{\frac{2}{3}}$. Prove that this sequence converges and find the limit.

Solution: First we claim that $a_n \leq 8$ for all n. Certainly this is true for the base case $n = 0$ since $a_0 = 1$. Suppose for the inductive step that $a_n \leq 8$. Then $a_{n+1} = 2a_n^{\frac{2}{3}} \leq 2(8)^{\frac{2}{3}} = 2(4) = 8$, as desired.

Next we claim that (a_n) is increasing, that is that $a_n \leq a_{n+1}$. Indeed, we do not need an induction for this step: if $a_n \leq 8$ then $a_n^{\frac{1}{3}} \leq 2$, so $a_{n+1} = 2a_n^{\frac{2}{3}} \geq a_n^{\frac{1}{3}} a_n^{\frac{2}{3}} = a_n$.

Since (a_n) is bounded above and increasing, it must converge to some limit $a > 1$. Applying the algebraic limit theorems to the relationship $a_n^3 = 8a_n^2$, we see that $a^3 = 8a^2$, or $a = 8$.

5. Let A and B be subsets of $\mathbb{R}$.

   (a) [3pts.] Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Solution: First suppose $x \in \overline{A} \cup \overline{B}$. If $x \in \overline{A}$, either $x \in A$ or x is a limit point of A. If $x \in A$, then clearly $x \in A \cup B$, hence $x \in \overline{A \cup B}$. If x is a limit point of A, there is a sequence of points (a_n) in A with $a_n \neq x$ for any n such that $\lim a_n = x$. Then (a_n) is also a sequence of points in $A \cup B$ with the same properties, hence we see that x is also a limit point of $A \cup B$, and therefore $x \in \overline{A \cup B}$. So $\overline{A} \subseteq \overline{A \cup B}$. Similarly $\overline{B} \subseteq \overline{A \cup B}$.

In the other direction, suppose that $x \in \overline{A \cup B}$. Again, if $x \in A \cup B$ then one of $x \in A$ and $x \in B$ is true, hence $x \in \overline{A} \cup \overline{B}$. Now suppose that x is a limit point of $A \cup B$. Then there is some sequence (c_n) of points in $A \cup B$ such that $c_n \neq x$ for any x and $\lim c_n = x$. Now, we see that $\{n : c_n \in A\} \cup \{n : c_n \in B\} = \mathbb{N}$, so at least one of these two sets of indices is infinite. Suppose without loss of generality that $\{n : c_n \in A\}$ is infinite. Then there is a subsequence (c_{n_k}) of (c_n) consisting of all the entries c_n of the sequence which lie in A. Since subsequences of a convergent sequence all converge to the limit of the sequence, we see that $\lim c_{n_k} = x$. Moreover, $c_{n_k} \neq x$ for all n_k. So x is a limit point of A and $x \in \overline{A} \subseteq \overline{A} \cup \overline{B}$. We conclude that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Since we now have inclusions in both directions, we have proved that $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

(b) [2pts.] What is the correct analog of this statement for closures of infinite unions?

Solution: The first half of the proof above is still true, but the step in the second half in which we claimed one set of indices was infinite no longer follows. For example, if $A_n = \{\frac{1}{n}\}$, then $\overline{A_n} = A_n$ and $\cup_{n=1}^{\infty} \overline{A_n} = \cup_{n=1}^{\infty} A_n = \{\frac{1}{n} : n \in \mathbb{N}\}$. But $\overline{\cup_{n=1}^{\infty} A_n} = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. So, the correct statement is

$$\cup_{n=1}^{\infty} \overline{A_n} \subseteq \overline{\cup_{n=1}^{\infty} A_n}.$$

Remark: The corresponding problem on the actual midterm is somewhat shorter.

6. Let (a_n) be a bounded sequence. The *limit supremum* of the sequence is

$$\limsup a_n = \lim_{N \to \infty} \sup\{a_n : n \geq N\}$$

(a) [1pts.] Show that the terms $u_N = \sup\{a_n : n \geq N\}$ are decreasing, and use this to conclude that the sequence above converges. Conclude that the limit supremum of a bounded sequence always exists.

Solution: Recall that if A and B are nonempty bounded above subsets of $\mathbb{R}$ with $A \subseteq B$, then $\sup A \leq \sup B$. Ergo, if $N > M$, we have that $u_M \leq u_N$. Thus this is a decreasing sequence which is bounded below since the sequence itself is bounded. It therefore converges.

(b) [1pts.] What is the limit supremum of the sequence (a_n) which begins $(2, 370, -5, 1, -1, 1, -1, 1, -1, \dots)$ and alternates between 1 and -1 thereafter?

Solution: Starting the indices at 1, we observe that for $N \geq 3$, we have $u_N = \sup\{a_n : n \geq N\} = 1$. Ergo the $\lim_{N\to\infty} u_N = 1$.

(c) [3pts.] Prove that the limit supremum is a subsequential limit of the sequence.

Solution: Let (a_n) be a bounded sequence and let $\epsilon > 0$. Say $t = \limsup a_n$, so that in the notation of part (a) we have $\lim_{N\to\infty} u_N = t$. There exists N_1 such that $u_{N_1} = \sup\{a_n : n > N_1\}$ is within distance $\epsilon_1 = 1$ of t; thus we have that $t \leq u_{N_1} < t + 1$. In particular, $1 - t$ is not an upper bound for $\{a_n : n \geq N_1\}$, so there exists n_1 such that $a_{n_1} \subseteq \{a_n : n \geq N_1\}$ and $1 - t < a_{n_1} < u_{N_1}$. We observe that $|a_{n_1} - t| < 1$.

Now, there exists $N_2 > n_1$ such that $u_{N_2} = \sup\{a_n : n > N_2\}$ is within distance $\epsilon_2 = \frac{1}{2}$ of t; thus we have that $t \leq u_{N_2} < t + \frac{1}{2}$. Repeating the argument above gives n_2 such that $n_1 < n_2$ and $|a_{n_2} - t| < \frac{1}{2}$. Iterating, we produce a subsequence of (a_n) converging to t.