Math 311H
Honors Introduction to Real Analysis

Sample Midterm 1

Instructions: You have 80 minutes to complete the exam. There are six questions, worth a total of thirty points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

RUID: ______________________________

Question	Points	Score
1	6	
2	4	
3	5	
4	5	
5	5	
6	5	
Total:	30	

1. For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.
   (a) [1pts.] A sequence which is not monotone and has no convergent subsequence.
   (b) [1pts.] A Cauchy sequence with a divergent subsequence.
   (c) [1pts.] A sequence with exactly two subsequential limits.
   (d) [1pts.] A bounded sequence with no Cauchy subsequence.
   (e) [1pts.] A bounded above nonempty subset A of $\mathbb{R}$ which contains its supremum but is not closed.
   (f) [1pts.] A series whose sum is 3.

2. [4pts.] Suppose that a_n and b_n are Cauchy sequences. Prove directly that $(a_n b_n)$ is a Cauchy sequence. ["Directly" here means that your proof should not reference the fact that Cauchy sequences converge in $\mathbb{R}$.]

3. [5pts.] Prove the *Root Test*: Let $\sum_{n=1}^{\infty} a_n$ be a series with the property that $\lim |a_n|^{\frac{1}{n}}$ exists and is equal to $L < 1$. Prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

4. [5pts.] Consider the sequence defined recursively by $a_0 = 1$ and $a_{n+1} = 2(a_n)^{\frac{2}{3}}$. Prove that this sequence converges and find the limit.

5. Let A and B be subsets of $\mathbb{R}$.
   (a) [3pts.] Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
   (b) [2pts.] What is the correct analog of this statement for closures of infinite unions?

6. Let (a_n) be a bounded sequence. The *limit supremum* of the sequence is

$$\limsup a_n = \lim_{N\to\infty} \sup\{a_n : n \geq N\}$$

(a) [1pts.] Show that the terms $u_N = \sup\{a_n : n \geq N\}$ are decreasing, and use this to conclude that the sequence above converges. Conclude that the limit supremum of a bounded sequence always exists.

(b) [1pts.] What is the limit supremum of the sequence (a_n) which begins $(2, 370, -5, 1, -1, 1, -1, 1, -1, \dots)$ and alternates between 1 and -1 thereafter?

(c) [3pts.] Prove that the limit supremum is a subsequential limit of the sequence.