Math 311H
Honors Introduction to Real Analysis

Midterm 1

Instructions: You have 80 minutes to complete the exam. There are six questions, worth a total of thirty points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

Question	Points	Score
1	6	
2	4	
3	5	
4	5	
5	5	
6	5	
Total:	30	

1. For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.
   (a) [1pts.] A Cauchy sequence with no monotone subsequence.
   (b) [1pts.] A monotone sequence with no Cauchy subsequence.
   (c) [1pts.] A sequence with exactly three subsequential limits.
   (d) [1pts.] A series $\sum_{n=1}^{\infty} a_n$ for which $\sum_{n=1}^{\infty} |a_n|$ converges but $\sum_{n=1}^{\infty} a_n$ does not.
   (e) [1pts.] An alternating series $\sum_{n=1}^{\infty} a_n$ of rational numbers converging to $\sqrt{2}$, and a partial sum of this series which is within .01 of $\sqrt{2}$.
   (f) [1pts.] A sequence of nonempty closed sets $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ such that $\bigcap_{n=1}^{\infty} F_n$ is empty.

2. (a) [3pts.] Let (a_n) be a sequence with the property that $|a_n - a_{n+1}| < \frac{1}{2^n}$. Show that (a_n) is Cauchy.

   (b) [1pts.] Is the conclusion of part (a) true if the condition is instead $|a_n - a_{n+1}| < \frac{1}{n}$?

3. Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \frac{5a_n}{3+a_n}$.
   (a) [3pts.] Prove that $1 \leq a_n \leq 2$ for all n and (a_n) is increasing.
   (b) [2pts.] Prove that (a_n) converges and compute the limit, justifying your steps carefully.

4. Let $\sum_{n=1}^{\infty} a_n$ be a series with the property that $\lim a_n = 0$.
   (a) [1pts.] Give an example to show that $\sum_{n=1}^{\infty} a_n$ need not necessarily converge.
   (b) [4pts.] Prove that there exists a subsequence (a_{n_k}) of (a_n) with the property that $\sum_{k=1}^{\infty} a_{n_k}$ converges. [Hint: Start by arguing that there is a subsequence (a_{n_k}) with the property that $|a_{n_k}| \leq \frac{1}{k^2}$.]

5. Let A_n be a subset of $\mathbb{R}$ for every n.
   (a) [3pts.] Prove that $\cup_{n=1}^{\infty} A_n^{\circ} \subseteq (\cup_{n=1}^{\infty} A_n)^{\circ}$.
   (b) [2pts.] Give an example to show the inclusion above may be proper.

6. A sequence (a_n) is said to *diverge to infinity* if, for every $M > 0$, there exists N such that $n \geq N$ implies that $a_n > M$.

   (a) [1pts.] Give an example (no need to justify it) of a sequence that diverges to infinity.

   (b) [1pts.] Give an example (no need to justify it) of a sequence that diverges but does not diverge to infinity.

   (c) [2pts.] Prove that if (a_n) is a sequence that diverges to infinity and (b_n) converges to a limit $b > 0$, then $(a_n b_n)$ also diverges to infinity.

   (d) [1pts.] Give an example to show that the statement above is not true if the condition $b > 0$ is removed.