Homework 9 Solutions

October 22, 2023

Section 3.3

0.1 Problem 3.3.5

(a) True. Let K and L be compact. Then since K and L are both closed, $K \cap L$ is closed. Moreover, since K and L are bounded, $K \cap L$ is bounded. Ergo, $K \cap L$ is closed and bounded, hence compact.

(b) False. Consider $K_n = \left[0, 1 - \frac{1}{n}\right]$. Then each K_n is a closed interval, hence compact, but $\bigcup_{n=1}^{\infty} K_n = [0, 1)$ is not closed, hence not compact.

(c) False. Let $K = [0, 3]$ and $A = (0, 1)$. Then K is compact but $K \cap A = A$ is not.

(d) False. Let $F_n = [n, \infty)$, which is closed. Then $F_1 \supset F_2 \subset F_3 \supset \dots$, but $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

Problem 3.3.8

Let K and L be compact, and let

$$d(K, L) = \inf\{|x - y| : x \in K, y \in :\}.$$

(a) We claim that if K and L are disjoint then $d(K, L) > 0$. For suppose not. Then for any $n \in \mathbb{N}$, we may find $x_n \in K$ and $y_n \in L$ such that $|x_n - y_n| < \frac{1}{n}$. Now, the sequence (x_n) is bounded since K is bounded, so there is some convergent subsequence (x_{n_m}) with $\lim x_{n_m} = x$. Because K is compact, $x \in K$. For any $\epsilon > 0$, choose M such that $\frac{1}{M} < \frac{\epsilon}{2}$ and $m \geq M$ implies that $|x_{n_m} - x| < \frac{\epsilon}{2}$. Now look at the corresponding subsequence (y_{n_m}) of y_n. For $m \geq M$ we have $|x - y_{n_m}| < |x - x_{n_m}| + |x_{n_m} - y_{n_m}| < \frac{\epsilon}{2} + \frac{1}{n_m} \leq \frac{\epsilon}{2} + \frac{1}{M} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So $y_{n_m} \to x$. Hence since x is closed, $x \in L$. But this is a contradiction, since K and L were supposed to be disjoint. So, $d(K, L) > 0$.
The second claim is very similar. Let $d = d(K, L)$, and for any $n \in \mathbb{N}$, choose $x_n \in K$ and $y_n \in L$ such that $|x_n - y_n| < d + \frac{1}{n}$. Pick a convergent subsequence (x_{n_k}) of (x_n) so that $x_{n_k} \to x \in K$. Then look a the corresponding sequence (y_{n_k}) and pick a convergent subsequence (y_{n_ℓ}) so that $y_{n_\ell} \to y$. Notice that $x_{n_\ell} \to x$ since subsequences of convergent sequences converge to the same limit. And furthermore $d \leq |x - y| \leq |x - x_{n_\ell}| + |x_{n_\ell} - y_{n_\ell}| + |y_{n_\ell} - y|$, which may be made less than $d + \epsilon$ for any ϵ by choosing ℓ sufficiently large. So $|x - y| = d$.

(b) Consider the sets $K = \mathbb{N}$ and $L = \left\{n + \frac{1}{2n} : n \in \mathbb{N}\right\}$, which are closed (neither of them has any limit points) but not compact. Then $0 \leq d(K, L) \leq \frac{1}{2n}$ for all $n \in \mathbb{N}$, so $d(K, L) = 0$.

Problem 3.3.12

Let A be a bounded infinite set. Suppose for the sake of contradiction that A has no limit points. Then, in particular, A is closed, hence compact. Now, for any point $a \in A$, we have that a is not a limit point of A, and therefore there is some neightborhood $O_a = V_{\epsilon_a}(a)$ containing no point of A other than A. The sets $\{O_a : a \in A\}$ are an open cover of A with no finite subcover, since each set O_a contains exactly one point of A. This contradicts compactness.

Section 3.4

3.4.1

Let P be perfect and K be compact. Then consider the intersection $P \cap K$. The intersection is not necessarily perfect; for example, we could take P to be $[0, 1]$ and K to be $\{0\}$, so that their intersection is the finite set $\{0\}$, which is not perfect. However, the intersection $P \cap K$ is always compact. For notice that P is in particular closed, so since K is closed, $P \cap K$ is closed. Furthermore since K is bounded, $P \cap K$ is bounded. So since $P \cap K$ is closed and bounded in $\mathbb{R}$, it is compact.

3.4.4

(a) We construct the fat Cantor set C' by removing the open middle quarter from each interval at each step, so that $C'_0 = [0, 1]$, $C'_1 = \left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right]$, and so on, and $C' = \bigcap_{n=0}^{\infty} C'_n$. Since this is the intersection of closed sets it is closed. Moreover, the endpoints of any given interval in any C'_n remain in C', so for any $x \in C'$, there is a point x_n of C' other than x with $|x - x_n| < \frac{1}{2^n}$ of x for any n, since we can always pick an endpoint of the interval in C_n containing x. (Here the $\frac{1}{2}$ comes from noting that at every stage the length of the intervals in C'_n is less than half the length of the intervals at the previous step.) So for any neighborhood $V_\epsilon(x)$, if we choose n such that $\frac{1}{2^n} < \epsilon$, we have $x_n \in V_\epsilon(x)$. So x is a limit point of C'. Since C' is closed and every point in C' is a limit point of C', we see C' is perfect.

3.4.7

(a) We claim $\mathbb{Q}$ is totally disconnected. For let $x < y$ in $\mathbb{Q}$. Find an irrational number a such that $x < a < y$, and let $A = (-\infty, a) \cap \mathbb{Q}$ and $B = (a, \infty) \cap \mathbb{Q}$. Then A and B are separated since neither has a limit point in the other, and $\mathbb{Q} = A \cap B$. Futhermore $x \in A$ and $y \in B$. Since x and y were arbitrary, $\mathbb{Q}$ is totally disconnected.

(b) The irrationals are also totally disconnected by the same argument; if $z < w$ are two irrationals we may cut at a rational number between them.