Homework 8 Solutions

October 18, 2023

Section 3.2

Problem 3.2.2

First we consider $A = \left\{(-1)^n + \frac{2}{n} : n \in \mathbb{N}\right\}$.

(d) Recall from last week that the limit points of A are -1 and 1. Since 1 is already in the set for $n = 2$, the closure of A is $\overline{A} = A \cup \{-1\}$.
Next we consider $B = \{x \in \mathbb{Q} : 0 < x < 1\}$.

(d) By part (a) from last week, the limit points of B are all of the points in $[0, 1]$ and there are no isolated points. Ergo, the closure of B is $\overline{B} = [0, 1]$.

Problem 3.2.10

(a) This is impossible. Suppose that A is a countable subset of $[0, 1]$. Then list the elements of $A = \{a_1, a_2, \dots\}$ such that $a_n \neq a_m$ if $n \neq m$. Consider the sequence (a_n). This sequence is bounded, since every element a_n is contained in $[0, 1]$. So by Bolzano-Weierstrass it has a convergence subsequence (a_{n_k}) with limit some a. The element a may appear in the subsequence (a_{n_k}), but it does so at most once, so we may delete it if it does. Then there is a sequence (a_{n_k}) of elements in A not equal to a converging to a. Therefore a is a limit point of A. Hence A must have at least one limit point.

(b) This is possible. Consider the set $B = \mathbb{Q} \cap [0, 1]$ consisting of all the rationals in the interval $(0, 1)$, which is certainly countable. Then see Problem 3.2.2 for the argument that every point of B is a limit point of B.

(c) This is impossible. Let A be a set with infinitely many isolated points. For each a and isolated point of A, there is some ϵ-neighborhood $V_\epsilon(a)$ containing no other points of A. Choose a rational number $r \in V_{\frac{\epsilon}{4}}(a)$ and a rational number s such that $\frac{\epsilon}{4} < s < \frac{\epsilon}{2}$. Then consider the neighborhood $V_s r$. Firstly we claim it contains a, since $|r - a| < \frac{\epsilon}{4} < s$. Secondly we claim it is contained in $V_\epsilon(a)$. For if $x \in V_s(r)$, then $|x - a| \leq |x - r| + |r - a| < s + \frac{\epsilon}{4} < \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon$. Ergo for each isolated point of A we have found a pair of rationals (s, r) such that $a \in V_s(r)$ and the neighborhood $V_s(r)$ contains no other element of A. But there are only countably many unique pairs of rational numbers. Hence, the number of isolated points of A is countable.

Problem 3.2.13

Suppose that $A \subset \mathbb{R}$ is a nonempty set which is both closed and open, and is not all of $\mathbb{R}$. Then we can find some $x \in \mathbb{R}$ such that $x \notin A$. Observe that $B = A \cap (-\infty, x) = A \cap [-\infty, x]$ is still both closed and open, since finite intersections of open sets are open and finite intersections of closed sets are closed. If B is nonempty, since B is bounded above it has a supremum in $\mathbb{R}$, call it y. Since B is closed, $y \in B$. Since B is open, $y \notin B$. This is a contradiction. If B is empty, we have A bounded below by x, and we may repeat this argument with the infimum of B, again obtaining a contradiction. So the original assumption that it was possible to find $x \notin A$ for A nonempty is false. Hence A is either $\emptyset$ or $\mathbb{R}$.

Problem 3.2.14

(a) Recall that $\overline{E}$ is the union of E and the set L of limit points of E. But E is closed if and only if E contains all its limit points, or equivalently if $L \subset E$ and therefore $\overline{E} = E \cup L = E$. So we are done.

Similarly, E° is the set of points $x \in E$ with the property that there is some $\epsilon > 0$ such that $V_\epsilon(x) \subseteq E$. But E is open if and only if every $x \in E$ has this property, or in other words if and only if $E^\circ = E$.

(b) Let $E \subseteq \mathbb{R}$. Since $\overline{E}$ is closed and contains E, $(\overline{E})^c$ is an open set contained in E^c. Therefore in particular, $(\overline{E})^c \subseteq (E^c)^\circ$. Now by the same token, $(E^c)^\circ$ is an open set contained in E^c, so $((E^c)^\circ)^c$ is a closed set containing $(E^c)^c = E$, hence contains $\overline{E}$. So $\overline{E} \subseteq ((E^c)^\circ)^c$, implying that $(E^c)^\circ \subseteq \overline{E}^c$. Ergo we see that $(\overline{E})^c = (E^c)^\circ$.

For the other statement, again start with $E \subset \mathbb{R}$. Let $F = E^c$. Then by the preceding part, $\overline{F}^c = (F^c)^\circ$, so we have that $\overline{E^c}^c = E^\circ$. Taking the complement of both sides we conclude that $\overline{E^c} = (E^\circ)^c$.

Section 3.3

Problem 3.3.1

Suppose that $K \subset \mathbb{R}$ is compact and nonempty. Then K is bounded, so K has a supremum and infimum. Moreover K is closed, and a closed bounded set contains its supremum and infimum, so K in fact contains its supremum and infimum.

Problem 3.3.2

(a) The set $\mathbb{N}$ is not compact. The sequence $(1, 2, 3, \dots)$ has no subsequence converging in $\mathbb{N}$, or indeed converging at all.

(b) The set $A = \mathbb{Q} \cap [0, 1]$ is not compact. The sequence $(.3, .31, .314, .3141, \dots)$ whose limit is $\frac{\pi}{10}$ has no subsequence converging in A.

(c) The Cantor set is compact, since it is closed and bounded in $\mathbb{R}$.

(d) The set $A = \left\{a_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$ is not compact. The sequence (a_n) has no subsequence converging in A.

(e) The set $A = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$ is compact. It is clearly bounded. As for whether it is closed, notice that a convergent sequence of points (a_n) in A has a convergent monotone subsequence (a_{n_k}). If (a_{n_k}) is not eventually constant, then after possibly deleting repeated terms it must be a subsequence of $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right)$, and therefore converge to 1. So 1 is the only limit point of A. And $1 \in A$, so in fact A is closed, hence compact since it is also bounded.

Problem 3.3.11

(a) The open cover $\left\{O_n = \left(n - \frac{1}{2}, n + \frac{1}{2}\right) : n \in \mathbb{N}\right\}$ of $\mathbb{N}$ has no finite subcover, since each O_n contains only a single point of the infinite set $\mathbb{N}$.

(b) Consider the sets

$$\begin{aligned} O_1 &= \mathbb{Q} \cap ((-1, .3) \cup (.4, 2)) \\ O_2 &= \mathbb{Q} \cap ((-1, .31) \cup (.32, 2)) \\ O_3 &= \mathbb{Q} \cap ((-1, .314) \cup (.315, 2)) \end{aligned}$$

and so on, so that O_n is missing all rationals within an interval of length $\frac{1}{10^n}$ containing $\frac{\pi}{10}$ but all rationals in the interval $[0, 1]$ fall within O_n for sufficiently large n. This open cover of $A = \mathbb{Q} \cap [0, 1]$ does not have a finite subcover - if it did, then since $O_1 \subset O_2 \subset O_3 \subset \dots$, we would have that $A \subset O_N$ for some N, which is plainly false.

(d) For the set $A = \left\{a_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$, we may consider the open cover $\left\{O_n = \left(a_n - \frac{1}{2(n+1)^2}, a_n + \frac{1}{2(n+1)^2}\right) : n \in \mathbb{N}\right\}$. Then each of the open sets O_n contains a single point of the infinite set A, hence there is no finite subcover.