Homework 7 Solutions

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Section 2.8

2.8.7

Let $\sum_{i=1}^{\infty} a_i$ converge absolutely to A and $\sum_{j=1}^{\infty} b_j$ converge absolutely to B.

(a) We wish to show $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges. By assumption $\sum_{j=1}^{\infty} |b_j|$ converges, to some C. Then for fixed i, we see that $\sum_{j=1}^{\infty} |a_i b_j| = \sum_{j=1}^{\infty} |a_i||b_j| = |a_i|C$ by the algebraic limit theorem. Now we have

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j| &= \sum_{i=1}^{\infty} |a_i| C \\ &= CD \end{aligned}$$

where D is the number such that $\sum_{i=1}^{\infty} |a_i| = D$. In particular the double sum converges.

(b) We observe that s_{nn} is the product of the nth partial sum t_n of $\sum a_i$ and r_n the nth partial sum of $\sum b_j$. Since $t_n \to A$ and $r_n \to B$, the Algebraic Limit Theorem tells us that $r_n t_n \to AB$. Now we recall from Theorem 2.8.1 that $\lim_{n\to\infty} s_{nn}$ is the double sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_i b_j$, which is precisely the sum $\sum_{k=1}^{\infty} d_k$ from the problem statement.

Section 3.2

Problem 3.2.2

First we consider $A = \left\{(-1)^n + \frac{2}{n} : n \in \mathbb{N}\right\}$. It is helpful to notice that we can rewrite A as the unions of the two sets $C = \left\{-1 + \frac{2}{n} : n \text{ odd } \right\} = \left\{-1 + \frac{2}{2n-1} : n \in \mathbb{N}\right\} = \left\{1, -\frac{1}{3}, -\frac{3}{5}, \dots\right\}$ and $D = \left\{1 + \frac{2}{n} : n \text{ even } \right\} = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\} = \left\{2, \frac{3}{2}, \frac{4}{3}, \dots\right\}$.

(a) We claim that the limit points of A are 1 and -1. To see that -1 is a limit point we observe that $(-1 + \frac{2}{2n-1})$ is a sequence of points in A not equal to -1 converging to -1; to see that 1 is a limit point we observe that $(1 + \frac{1}{n})$ is a sequence of points in A not equal to 1 converging to 1. Now we must show that there are no other limit points. For $x \in \mathbb{R}$ which is not 1 or -1, let $\epsilon = \frac{1}{2}\min\{|x-1|, |x-(-1)|\}$. Then in particular $V_\epsilon(x)$ has no intersection with $V_\epsilon(1)$ and $V_\epsilon(-1)$. Now, for $N > \frac{2}{\epsilon}$, all of the points $(-1)^n + \frac{2}{n}$ for which n is odd lie in $V_\epsilon(-1)$ and all of the points $(-1)^n + \frac{2}{n}$ for which n is even lie in $V_\epsilon(1)$. In particular $V_\epsilon(x)$ contains finitely many (at most $2N-2$) points of A. But if x were a limit point of A, every neighborhood of x would contain infinitely many points of A. Ergo, x is not a limit point of A.

(b) The set is not open; observe that any neighborhood $V_\epsilon(2)$ contains a point x such that $x > \frac{3}{2}$, which therefore is not in A. So there is no $\epsilon > 0$ such that $V_\epsilon(2)$ lies in A. Hence A is not open. The set A is also not closed, since -1 is a limit point not contained in A.

(c) All of the points of A except for 1 are isolated points, by the argument in part (a).

Next we consider $B = \{x \in \mathbb{Q} : 0 < x < 1\}$.

(a) The set of limit points of B is the entire closed interval $[0, 1]$. For let $q \in [0, 1]$. Then we claim that any ϵ-neighborhood $V_\epsilon(q)$ contains a rational number not equal to q in $(0, 1)$. For if $q \neq 0, 1$, there is some $\epsilon' < \epsilon$ such that $V_{\epsilon'}(q) \subset (0, 1)$. Then there is a rational number $r \neq q$ in the interval $(q - \epsilon', q)$ which is an element of $(0, 1)$ and therefore of B, and $r \in (q - \epsilon', q) \subset V_{\epsilon'}(q) \subset V_\epsilon(q)$, so every neighborhood of q contains a point of B other than q and q is therefore a limit point of A. Likewise, if $q = 0$, there is some $\epsilon' < \epsilon$ such that $(0, \epsilon') \subset (0, 1)$, and there is a rational number $r \neq q$ in $(0, \epsilon')$ which therefore also lies in $V_\epsilon(0)$, so every neighborhood of 0 contains a point of B, hence 0 is a limit point of B. Similarly 1 is a limit point of B. We conclude that the limit points of B are $[0, 1]$.

(b) The set is neither open nor closed. For not open, observe that every ϵ-neighborhood $V_\epsilon(\frac{1}{2})$ contains an irrational number, hence is not a subset of B. So $\frac{1}{2}$ has no ϵ-neighborhood which is a subset of B. For not closed, observe that 1 is a limit point of B not contained in B.

(c) There are no isolated points; every point is a limit point, as discussed in part (a).

Problem 3.2.3

(a) The set $\mathbb{Q}$ is neither open nor closed. To see that it is not open, consider $0 \in \mathbb{Q}$. Any $V_\epsilon(0) = (-\epsilon, \epsilon)$ contains an irrational number. Thus there does not exist any ϵ such that $V_\epsilon(0) \subset \mathbb{Q}$. To see that the set is not closed, consider $\sqrt{2}$. Any neighborhood $V_\epsilon(\sqrt{2})$ contains a rational number, so $\sqrt{2}$ is a limit point of $\mathbb{Q}$ not contained in $\mathbb{Q}$. Hence $\mathbb{Q}$ is not closed.

(b) The set $\mathbb{N}$ is not open, but is closed. To see that it is not open, consider $0 \in \mathbb{N}$. For any $\epsilon > 0$, $V_\epsilon(0)$ contains a point that is not a natural number, hence is not contained in $\mathbb{N}$. So $\mathbb{N}$ is not open. To see that it is closed, notice that every element $n \in \mathbb{N}$ has a neighborhood $(n - \frac{1}{2}, n + \frac{1}{2})$ which contains no other element of $\mathbb{N}$. So every element of $\mathbb{N}$ is an isolated point. Moreover, any $x \notin \mathbb{N}$ has a neighborhood $V_\epsilon(x)$ containing no natural numbers by letting $n < x < n + 1$ and setting $\epsilon = \min\{|x - n|, |x - (n + 1)|\}$. So x is not a limit point of $\mathbb{N}$. Ergo $\mathbb{N}$ has no limit points, hence trivially contains all its limit points and is closed.

(c) The set $A = \{x \in \mathbb{R} : x \neq 0\}$ is open but not closed. To see it is open, observe that $A = (-\infty, 0) \cup (0, \infty)$ is the union of two open intervals, hence open. To see it is not closed, observe that for any $\epsilon > 0$, the neighborhood $V_\epsilon(0)$ contains a point of A. So 0 is a limit point of A not contained in A, hence A is not closed.

(d) The set $A = \left\{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$ is neither closed nor open. For not open, observe that any neighborhood $V_\epsilon(1)$ contains a point not in A. For not closed, let $\alpha = \sum_{n=1}^{\infty} \frac{1}{n^2}$. Then there is a sequence of points (a_n) in A with $a_n \neq \alpha$ converging to α, by taking $a_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$. So α is a limit point of A not contained in A. Hence A is not closed.

(e) The set $A = \left\{a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} : n \in \mathbb{N}\right\}$ is closed but not open. For not open, observe that any neighborhood $V_\epsilon(1)$ contains a point not in A. For not closed, suppose for the sake of contradiction that α is a limit point of A. Then there is a sequence of points (b_n) in A converging to α which does not contain α as any of its elements, hence is not eventually constant. By Bolzano-Weierstrass, the sequence (b_n) must have a monotone subsequence (b_{n_k}), which also converges to α because subsequences of convergent sequences all converge to the same limit, and is also not eventually constant. However, (a_n) is already in monotone increasing order, which implies that after possibly deleting repeated terms, (b_{n_k}) appears as some subsequence (a_{n_ℓ}) of (a_n). However, (a_n) is monotone and unbounded above, hence does not have any convergent subsequences. This is a contradiction. So A is closed.

Problem 3.2.6

(a) False; consider $(\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$.

(b) False; let $A_n = [n, \infty)$ for $n \in \mathbb{N}$. Each A_n is closed and $A_1 \supset A_2 \supset A_3 \supset \ldots$, but $\cap_{n=1}^{\infty} A_n = \emptyset$.

(c) True. Let A be open and nonempty. Choose an element $a \in A$. Then there is some ϵ such that $V_\epsilon(a) \subseteq A$, and there is certainly a rational number in $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$. So A contains a rational number.

(d) False. Consider $A = \{\sqrt{2}\} \cup \{\sqrt{2} + \frac{1}{n} : n \in \mathbb{N}\}$. This set A has a single limit point, $\sqrt{2}$, which it contains; therefore A is closed. The set A is also clearly infinite, and is bounded, since for all $a \in A$, $0 < a < 4$.

(e) True. The Cantor set C is constructed as an intersection of sets C_i. Each C_i is the finite union of closed intervals, hence closed since the finite union of closed sets is closed. And then $C = \cap_{i=1}^{\infty} C_i$ is the infinite intersection of closed sets, hence closed since intersections of closed sets are closed.

Other Problems

Problem 6

(a) Let $b_n = \frac{1}{n!}$. Then $|\frac{b_{n+1}}{b_n}| = |\frac{1}{n+1}|$, so $\lim |\frac{b_n+1}{b_n}| = 0$. Ergo $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{n!}$ converges absolutely by the Ratio Test.

(b) We compute that

$$\begin{aligned}
a_n &= (1+\frac{1}{n})^n \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\left(\frac{1}{n}\right)^k \\
&= \sum_{k=0}^{n} \frac{n!}{(n-k)!}\left(\frac{1}{n}\right)^k \frac{1}{k!} \\
&= \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{1}{k!}
\end{aligned}$$

Observe that for $n \geq 1$, $\frac{n(n-1)\cdots(n-k+1)}{n^k} \leq 1$, so $a_n \leq \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!} = s_n$. Note that since (s_n) is increasing, this in particular implies $a_n \leq s$ for all $n \geq 1$.

(c) Notice that

$$\begin{aligned}
a_n &= \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{1}{k!} \\
&= \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{1}{k!} + \sum_{k=m+1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{1}{k!} \\
&\geq \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{1}{k!}
\end{aligned}$$

Fix m and call the righthand side t_n^m. Then as $n \to \infty$, we see that $t_n^m \to 1+1+\frac{1}{2!}+\cdots\frac{1}{m!} = s_m$. In particular, given any ϵ, we observe that there exists some N_m such that $n \geq N_m$ implies that $t_n^m > s_m - \frac{\epsilon}{2}$. Ergo $n \geq N_m$ implies in particular that $a_n > s_m - \frac{\epsilon}{2}$.

(d) Now we complete the proof. Let $\epsilon > 0$. Choose any integer m such that $s - \frac{\epsilon}{2} < s_m \leq s$, which certainly exists since the sequence (s_n) converges to s. Then choose N_m as in part (c) so that $n \geq N_m$ implies $a_n > s_m - \frac{\epsilon}{2}$. Then in total we have

$$s \geq a_n > s_m - \frac{\epsilon}{2} > s - \frac{\epsilon}{2} - \frac{\epsilon}{2} = s - \epsilon.$$

In particular $n \geq N_m$ implies $|a_n - s| < \epsilon$. So $\lim a_n = s$.
This outline is based on the proof given in Rudin's book *Principles of Mathematical Analysis*.