MATH 311H: Homework 7

Due: October 23 at 5 pm

1. Upcoming office hours are Monday and Thursday 10-11 in LSH 102D.
2. A reminder that Midterm 1 is on Thursday, October 26. It will be six questions and you will have the full period to work on it. It will cover through the end of Section 3.2 in Abbott, which means it will cover through close to the end of lecture on Thursday, October 19. A sample midterm will be posted this week. The final question on the midterm will ask you to consider a definition we haven't seen yet and answer some simple questions about it. (The version of this question on the sample midterm is slightly more difficult than the one on the actual midterm.)
3. Warning: Most people find the exercises from Chapter 3 a little time-consuming to think through the first time they see them. It is not recommended to do this set at the last minute.
4. Read Sections 3.2-3 in Abbott.
5. Do Abbott Exercise 2.8.7*, 3.2.3*, and 3.2.6.
6. Do Abbott Exercise 3.2.2* parts (a)-(c) in Abbott. [We won't have quite covered enough material for part (d) as of the end of Monday; it will be on the homework next week.]
7. *The number e.* You have probably seen in calculus that Euler's number e may be defined as the limit of the sequence $a_n = (1 + \frac{1}{n})^n$. This is sometimes described as the interaction between the "irresistible force" – to wit, an exponent approaching infinity – and the "immovable object" – to wit, a base approaching 1. Another possible definition of e is

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

We will show these expressions are both convergent, and in fact coincide. Let s_n be the partial sums of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$.

- (a) Show that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. Call the limit s.
- (b) The binomial theorem states that, for $n \geq 1$, $(1+x)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k$. With this in mind, show that for $n \geq 1$,

$$a_n = \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!}$$

  Conclude that $a_n \leq s_n$ for all $n \geq 1$. Note that since (s_n) is increasing, this in particular implies $a_n \leq s$ for all $n \geq 1$.
- (c) For $n \geq m$, show that

$$a_n \geq \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!}.$$

Fix m and call the righthand side t_n^m. Then as $n \to \infty$, we see that $t_n^m \to 1 + 1 + \frac{1}{2!} + \cdots \frac{1}{m!} = s_m$. In particular, given any ϵ, we observe that there exists some N_m such that $n \geq N_m$ implies that $t_n^m > s_m - \frac{\epsilon}{2}$. Ergo $n \geq N_m$ implies in particular that $a_n > s_m - \frac{\epsilon}{2}$.

- (d) Now we complete the proof. Let $\epsilon > 0$. Choose any integer m such that $s - \frac{\epsilon}{2} < s_m \leq s$, which certainly exists since the sequence (s_n) converges to s. Then choose N_m as in part (c) so that $n \geq N_m$ implies $a_n > s_m - \frac{\epsilon}{2}$. Then in total we have

$$s \geq a_n > s_m - \frac{\epsilon}{2} > s - \frac{\epsilon}{2} - \frac{\epsilon}{2} = s - \epsilon.$$

  In particular $n \geq N_m$ implies $|a_n - s| < \epsilon$. So $\lim a_n = s$.