Homework 6 Solutions

October 17, 2023

Section 2.5

Problem 2.5.5

Let (a_n) be a bounded sequence with the property that every convergent subsequence converges to the same limit a. Suppose, for the sake of inducing a contradiction, that (a_n) does not converge to a. Then there is some $\epsilon > 0$ such that for any N a natural number, we can find $n > N$ such that $|a_n - a| \geq \epsilon$. In particular, we may pick a subsequence (a_{n_k}) of (a_n) consisting of terms of distance at least ϵ from a. Now, (a_{n_k}) is a bounded sequence, hence by Bolzano-Weierstrass it has a convergent subsequence (a_{n_j}) with limit some real number b. However, (a_{n_j}) consists solely of points of distance at least ϵ from a, so $|b - a| \geq \epsilon$ and in particular $b \neq a$. So (a_{n_j}) is a subsequence of (a_n) not converging to a. This is a contradiction. So $\lim a_n = a$.

Problem 2.5.6

We wish to compute the limit of $(b^{\frac{1}{n}})$ for all $b \geq 0$. If $b = 0$ this limit obviously exists and is 0; similarly if $b = 1$ the limit obviously exists and is 1.

Now consider the case $0 < b < 1$. We see first that $c_n = (b^{\frac{1}{n}})$ is positive and bounded above by 1. Moreover we claim the sequence is increasing. In particular, $c_n^n = b = c_{n+1}^{n+1}$. Since $c_{n+1} < 1$, this implies that $c_{n+1}^n > b = c_n^n$, which in turn implies that $c_{n+1} > c_n > b$. So the sequence is bounded monotone, hence convergent. Let the limit be ℓ. Observe that $b \leq \ell \leq 1$ by the order limit theorem. We consider the subsequence $(b^{\frac{1}{2n}})$. By the results of last week's homework, this subsequence converges to $\sqrt{\ell}$. But every subsequence of a convergent sequence converges to the limit of the sequence, so in fact we must have $\ell = \sqrt{\ell}$. This implies $\ell = 1$. So $\lim b^{\frac{1}{n}} = 1$.

The case that $b > 1$ follows by writing $b = \frac{1}{c}$ so that $b = \frac{1}{c^{\frac{1}{n}}}$ and applying the algebraic limit theorem to the quotient. In particular we see that $\lim b^{\frac{1}{n}} = 1$ again.

Section 2.6

Problem 2.6.2

(a) The sequence $(\frac{(-1)^n}{n})$ is convergent, hence Cauchy, but not monotone.

(b) Cauchy sequences are bounded, so every subsequence of a Cauchy sequence is bounded. Hence this is impossible.

(c) Impossible; we claim that a monotone sequence with a Cauchy subsequence must be bounded, hence convergent. To justify this, suppose (a_n) is increasing with a Cauchy subsequence (a_{n_k}). Since Cauchy sequences are bounded there is some M such that $a_{n_k} < M$ for all M. Then for any n, there is some $n_k > n$, so $a_n \le a_{n_k} \le M$, so (a_n) is also bounded by M. The case for descreasing sequences is similar.

(d) Consider the sequence $(1, 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \dots)$, which is unbounded but contains the Cauchy sequence $\left(\frac{1}{n}\right)$ as a subsequence.

Problem 2.6.4

Let (a_n) and (b_n) be Cauchy sequences.

(a) Yes, (c_n) where $c_n = |a_n - b_n|$ is Cauchy. Given $\epsilon > 0$, there is some N_1 such that $n, m \ge N_1$ implies $|a_n - a_m| < \frac{\epsilon}{2}$ and some N_2 such that $n, m \ge N_2$ implies that $|b_n - b_m| < \frac{\epsilon}{2}$. Then for $n, m \ge N = \max\{N_1, N_2\}$, we have

$$\begin{aligned} |a_n - b_n| &\le |a_n - a_m| + |a_m - b_m| + |b_m - b_n| \\ &< |a_m - b_m| + \epsilon. \end{aligned}$$

and similarly $|a_m - b_m| < |a_n - b_n| + \epsilon$, so in total $|a_m - b_m| - \epsilon < |a_n - b_n| < |a_m - b_m| + \epsilon$. In particular $n, m \ge N$ implies $||a_m - b_m| - |a_n - b_n|| < \epsilon$.

(b) The sequence $((-1)^n a_n)$ need not be Cauchy. For example, $a_n = 1$ is a Cauchy sequence but $((-1)^n a_n)) = ((-1)^n)$ is not.

(c) The sequence $([[a_n]])$ need not be Cauchy; consider the Cauchy sequence $a_n = \left(\frac{(-1)^n}{n}\right)$, which has $([[a_n]]) = (-1, 0, -1, 0, \dots)$.

1 Section 2.7

Problem 2.7.2

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ converges by observing that $\frac{1}{2^n+n} < \frac{1}{2^n}$ and applying the Comparison Test.

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges by the Comparison Test since $\left|\frac{\sin(n)}{n^2}\right| \le \frac{1}{n^2}$.

(c) Diverges; notice that the absolute values of the terms are $\frac{n+1}{2n}$ which converges to $\frac{1}{2}$ rather than 0.

(d) Diverges. Let (s_m) be the partial sums of the series. Notice that $s_{3k} > 1 + \frac{1}{4} + \frac{1}{7} + \dots + \frac{1}{3k-2}$. If t_m are the partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{3n-2}$, then $s_{3k} = t_k$. And $\sum_{n=1}^{\infty} \frac{1}{3n-2}$ clearly diverges, for example by applying limit comparison to $\sum \frac{1}{n}$, so the terms t_k are unbounded above, implying that the partial sums s_{3k} are unbounded above and our original series diverges.

(e) Diverges. Let (s_m) be the partial sums of the series, (t_m) be the sums of the divergent series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ and (r_m) be the sums of the convergent series $\sum_{n+1}^{\infty} \frac{1}{(2n)^2}$. (Both of these assertions can be quickly confirmed by limit comparison to the obvious thing.) Then the sums t_m are unbounded above and the sums r_m are bounded above, say by some M. We have that $s_{2m} = t_m - r_m$. Given any natural number N, choose m such that $t_m > N + M$, so that $s_{2m} = t_m - r_m > N$. This shows the partial sums s_{2m} are not bounded above. We conclude the series diverges.

Problem 2.7.8

(a) True. Suppose $\sum a_n$ converges absolutely, so that $\sum |a_n|$ converges. Then $a_n \to 0$. There is therefore some N such that $n \geq N$ implies that $|a_n| < 1$. Then for $n \geq N$, we have that $|a_n|^2 < |a_n|$, implying by the Comparison Test that $\sum |a_n|^2$ converges.

(b) False. Consider $\sum a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ and $(b_n) = \left(\frac{(-1)^{n+1}}{\sqrt{n}}\right)$. It's true if you assume absolute convergence, though: (b_n) is bounded by some M and $|a_n b_n| \leq M|a_n|$, so $a_n b_n$ converges by comparison if $\sum |a_n|$ converges.

(c) True. For suppose that $\sum n^2 a_n$ converges. Then $n^2 a_n \to 0$, so there is some N such that $n \geq N$ implies that $|n^2 a_n| < 1$, or in particular $|a_n| < \frac{1}{n^2}$. This implies that $\sum a_n$ converges absolutely. So if $\sum a_n$ converges conditionally, it must be the case that $\sum n^2 a_n$ diverges.

Other Problems

Problem 4

(a) $\{1, 5\}$.

(b) $\{0, \pm\frac{\sqrt{3}}{2}\}$

(c) $\{0\}$

(d) $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$

(e) All of $\mathbb{R}$.

Problem 5

(a) Suppose that $a_1, \cdots a_{n-1}$ have been chosen as specified, and that a_n is the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_n}{k^n} \leq x$$

First suppose that $a_n < 0$. This implies that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{0}{k^n} > x$$

which is impossible by construction of a_{n-1}. So $a_n \geq 0$. Now suppose $a_n \geq k$. Then $\frac{a_n}{k^n} > \frac{k}{k^n} = \frac{1}{k^{n-1}}$. This implies that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_{n-1}}{k^{n-1}} + \frac{1}{k^{n-1}} \leq x$$

so in particular

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_{n-1}+1}{k^{n-1}} \leq x$$

which is impossible, since a_{n-1} was chosen to be the largest integer such that the equation above was satisfied. So $a_n \leq k-1$. Hence $0 \leq a_n \leq k-1$ for $i \geq 1$.

(b) First, x is clearly an upper bound for $\{r_0, r_1, \dots\}$. Suppose $y < x$. Then we may choose M such that $x - y > \frac{1}{k^M}$. Now consider

$$r_M = a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_M}{k^M} \leq x.$$

Since a_M is the largest integer for the inequality above is true, we see that $x - r_M \leq \frac{1}{k^M}$. This implies that $r_M > y$. So y is not an upper bound for $\{r_0, r_1, \cdots\}$. Therefore since no number less than x is an upper bound for $\{r_0, r_1, \cdots\}$, it follows that $x = \sup\{r_0, r_1, \cdots\}$.

Now, we observe that the partial sums of $\sum_{n=0}^{\infty} \frac{a_n}{k^n}$ are the increasing sequence (r_n), bounded above by x. As an increasing bounded sequence converges to the supremum of its terms, we have that $\sum_{n=0}^{\infty} \frac{a_n}{k^n} = x$.

(c) Let $r_0 = 0$ and for $n > 0$ let $r_n = 0 + \frac{k-1}{k} + \frac{k-1}{k^2} + \cdots + \frac{k-1}{k^n}$. We claim that $\sup\{r_0, r_1, \dots\}$ is equal to 1. For certainly $r_n < 1$, so 1 is an upper bound of the set. Furthermore, if $y < 1$, we may choose M so that $\frac{1}{k^M} < 1 - y$ so that, since $1 - r_M = \frac{1}{k^M}$, we have $y < r_M < 1$. So, indeed, y is the supremum of the set, and $\sum_{n=1}^{\infty} \frac{k-1}{k^n} = 1$.

(d) Let $S = \{r_0, r_1, \dots\}$ and $S' = \{r'_0, r'_1, \dots\}$. Let $x = \sup S = \sup S'$. Suppose that $a_0 \neq a'_0$. Without loss of generality we may assume $a_0 < a'_0$. We observe that

$$\begin{aligned}
r_n &= a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_n}{k^n} \\
&\leq a_0 + \frac{k-1}{k} + \cdots + \frac{k-1}{k^{n-1}} + \frac{k-1}{k^n} \\
&< a_0 + \frac{k-1}{k} + \cdots + \frac{k-1}{k^{n-1}} + \frac{k}{k^n} \\
&= a_0 + \frac{k-1}{k} + \cdots + \frac{k-1}{k^{n-2}} + \frac{1}{k^{n-1}} \\
&= a_0 + 1
\end{aligned}$$

So in particular $a_0 + 1$ is an upper bound for $\{r_0, r_1, \cdots\}$, implying that $x \leq a_0 + 1$. Since $r'_0 = a'_0$ is an element of S' and $x = \sup S'$, we see that $a'_0 = r'_0 \leq x \leq a_0 + 1$. Since we are assuming $a_0 \neq a'_0$, we must have $a'_0 = a_0 + 1$. So in fact $a_0 + 1 = a'_0 \leq x \leq a_0 + 1$, implying that $x = a'_0 = a_0 + 1$.

However, recall that by assumption, there is some $i \geq 1$ such that $a_i < k - 1$. Choose the smallest such i. Then we have

$$r_i = a_0 + \frac{k-1}{k} + \frac{k-1}{k^2} + \cdots + \frac{k-1}{k^{i-1}} + \frac{a_i}{k^i}$$

By the same argument as above, for $n > i$,

$$\begin{aligned} r_n &< a_0 + \frac{k-1}{k} + \frac{k-1}{k^2} + \cdots + \frac{k-1}{k^{i-1}} + \frac{a_i}{k^i} + \frac{1}{k^i} \\ &= a_0 + \frac{k-1}{k} + \frac{k-1}{k^2} + \cdots + \frac{k-1}{k^{i-1}} + \frac{a_i+1}{k^i} \\ &< a_0 + 1 - \frac{k-(a_i+1)}{k^i} \end{aligned}$$

This implies that $x = \sup S$ has $x \leq a_0 + 1 - \frac{k-(a_i+1)}{k^i} < a_0 + 1$. This is a contradiction. So in fact $a_0 = a_0'$. Repeating the argument proves that $a_i = a_i'$.