# MATH 311H: Homework 6

Due: October 16 at 5 pm

1. Upcoming office hours are Monday October 9 and Thursday October 12 10-11 am in LSH B-102D.
2. Read Sections 3.1-2 in Abbott.
3. Do Abbott Exercises 2.5.5, 2.5.6*, 2.6.2*, 2.6.4, 2.7.2, 2.7.8
4. For each of the sequences below, what is the set of subsequential limits? (You don't have to prove your answer, just state it.)
   (a) $a_n = 3 + 2(-1)^n$
   (b) $b_n = \sin(\frac{n\pi}{3})$
   (c) $c_n = n\cos(\frac{n\pi}{4})$
   (d) $d_n = \left(1, \frac{1}{2}, 1, \frac{1}{3}, \frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, \dots\right)$
   (e) $(r_n)$ the enumeration of the rationals constructed in class.
5. *Existence and near-uniqueness of decimal expansions in arbitrary base.** Given $x \in \mathbb{R}$ with $x > 0$ and an integer $k \geq 2$, define $a_0, a_1, a_2, \dots$ recursively by setting $a_0$ to be the largest integer less than or equal to $x$ and $a_n$ to be the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_n}{k^n} \leq x.$$

   (a) Show that $0 \leq a_i \leq k - 1$ for all $i \geq 1$.
   (b) Let $r_n = a_0 + \frac{a_1}{k} + \cdots + \frac{a_n}{k^n}$. Show that $\sup\{r_0, r_1, \dots\} = x$, and use this to conclude that $\sum_{n=0}^{\infty} \frac{a_k}{k^n} = x$.
   (c) Confirm that $\sum_{n=1}^{\infty} \frac{k-1}{k^n} = 1$. This is the general case of .9999... = 1.
   (d) Show that if we have sequences of integers $(a_0, a_1, \dots)$ and $(a_0', a_1', \cdots)$ such that
   - $0 \leq a_i \leq k - 1$ and $0 \leq a_i' \leq k - 1$ for all $i$.
   - If $r_1$ and $r_i'$ are the sums defined in part (b) for $a_i$ and $a_i'$ respectively, then $\sup\{r_0, r_1, \dots\} = \sup\{r_0', r_1', \dots\}$.
   - For each $N$ there exists $n > N$ and $m > N$ such that $a_n \neq k - 1$ and $a_m' \neq k - 1$.

   then $a_i = a_i'$ for all $i$. This shows uniqueness of decimal expansions apart from trailing strings of $k - 1$'s.

   Hint: For the last part, it suffices to give an argument that $a_0 = a_0'$; after that, the logic for each successive term follows by induction.