Homework 5 Solutions

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Section 2.3

2.3.1

(a) Let $x_n \geq 0$, and suppose that $x_n \to 0$. For $\epsilon > 0$, there is a natural number N such that $n \geq N$ implies that $|x_n - 0| = x_n < \epsilon^2$. So, $n \geq N$ implies that $\sqrt{x_n} < \epsilon$, and in particular that $|\sqrt{x_n} - 0| < \epsilon$. Hence $\sqrt{x_n} \to 0$.

(b) Let $x_n \to x$. Observe that $x_n - x = (\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})$. So we have

$$\sqrt{x_n} - \sqrt{x} = (x_n - x)\frac{1}{\sqrt{x_n} + \sqrt{x}}$$

First choose N_1 so that $n \geq N_1$ implies that $\frac{x}{4} < x_n$ by setting $\epsilon_0 = \frac{3x}{4}$ in the definition of convergence. Then for $n \geq N_2$ we have $\frac{\sqrt{x}}{2} < \sqrt{x_n}$, and in particular $\frac{3}{2}\sqrt{x} < \sqrt{x_n} + \sqrt{x}$. Now given $\epsilon > 0$, choose N_2 such that $n \geq N_2$ implies that $|x_n - x| < \frac{3}{2}\sqrt{x}\epsilon$. Then we have that $n \geq \max\{N_1, N_2\}$ implies that

$$|\sqrt{x_n} - \sqrt{x}| = |x_n - x|\left|\frac{1}{\sqrt{x_n} + \sqrt{x}}\right| < \frac{3}{2}\sqrt{x}\epsilon\left(\frac{1}{\frac{3}{2}\sqrt{x}}\right) = \epsilon.$$

2.3.3

We have $x_n \leq y_n \leq z_n$ with $\lim x_n = \lim z_n = \ell$. Note that we can't use the order limit theorem directly because we don't yet know that y_n converges. However, let $\epsilon > 0$. There exists N_1 such that $n \geq N_1$ implies that $|x_n - \ell| < \epsilon$ and in particular that $\ell - \epsilon < x_n$. Furthermore there exists N_2 such that $n \geq N_2$ implies that $|z_n - \ell| < \epsilon$, and in particular $z_n < \ell + \epsilon$. We see that for $n \geq N = \max\{N_1, N_2\}$ we have

$$\ell - \epsilon < x_n \leq y_n \leq z_n \leq \ell + \epsilon$$

and in particular $|x_n - \ell| < \epsilon$. So $x_n \to \ell$.

2.3.5

Given sequences (x_n) and (y_n), we take (z_n) to be the shuffled sequence $(x_1, y_1, x_2, y_2, \dots)$ with $z_{2n-1} = x_n$ and $z_{2n} = y_n$. First assume that $\lim x_n = a = \lim y_n$. Then for any $\epsilon > 0$ there exists N_1 such that $n \geq N_1$ implies that $|x_n - a| < \epsilon$ and N_2 such that $n \geq N_2$ implies that $|y_n - a| < \epsilon$. Now let $n \geq N = \max\{2N_1, 2N_2\}$. If n is odd, $z_n = x_{\frac{n+1}{2}}$, so since $\frac{n+1}{2} \geq \frac{2N_1+2}{2} > N_1$, we have that $|z_n - a| = |x_n - a| < \epsilon$. Similarly if n is even $z_n = y_{\frac{n}{2}}$, so since $\frac{n}{2} \geq \frac{2N_2}{2} = N_2$, we have that $|z_n - a| = |y_{\frac{n}{2}} - a| < \epsilon$. So in either case $n \geq N$ implies that $|z_n - a| < \epsilon$, and we conclude that $z_n \to a$.

Now assume that $\lim z_n = a$ for some a. Then for any $\epsilon > 0$ there exists N such that $n \geq N$ implies that $|z_n - a| < \epsilon$. Now for $n \geq N$, observe that $x_n = z_{2n-1}$. If $n \geq N$, we also have that $2n - 1 \geq N$, so for $n \geq N$ we have $|x_n - a| = |z_{2n-1} - a| < \epsilon$. Hence $x_n \to a$. Likewise if $n \geq N$ then $|y_n - a| = |z_{2n} - a| < \epsilon$, so $y_n \to a$.

2.3.10

(a) False! Let $a_n = (-1)^n$, $b_n = (-1)^n$. Both (a_n) and (b_n) diverge, but their difference $a_n - b_n = 0$ is a constant sequence converging to 0.

(b) True. Suppose that $b_n \to b$. Let $\epsilon > 0$, then there exists N such that $n \geq N$ implies that $|b_n - b| < \epsilon$. Now by the Triangle Inequality we have

$$|b_n| = |(b_n - b) + b| \leq |b_n - b| + |b|$$

so in particular $|b_n| - |b| \leq |b_n - b|$. By the same logic $|b| - |b_n| \leq |b_n - b|$. Therefore we see that $||b_n| - |b|| \leq |b_n - b|$. In particular, if $n \geq N$, we have $||b_n| - |b|| \leq |b_n - b| < \epsilon$. Ergo $|b_n| \to |b|$.

(c) True. Let $\lim a_n = a$ and $\lim b_n - a_n \to 0$. Then by the algebraic limit theorem, $\lim b_n = \lim(b_n - a_n + a_n) = \lim(b_n - a_n) + \lim a_n = 0 + a = a$.

(d) True. Let $a_n \to 0$. Then for any $\epsilon > 0$, there exists N such that $n \geq N$ implies that $|a_n| < \epsilon$. So for $n \geq N$, we have $|b_n - b| < |a_n| < \epsilon$. We conclude that $b_n \to b$.

1 Section 2.4

2.4.1

(a) Let $x_1 = 3$ and $x_{n+1} = \frac{1}{4 - x_n}$. First we claim that $0 \leq x_n \leq 3$ for all n. This is clearly true for the base case $x_1 = 3$. For the inductive step, suppose we know that $0 \leq x_n \leq 3$. Then since $x_n \leq 3$, we have that $4 - x_n$ is positive, so $x_{n+1} = \frac{1}{4-x_n} > 0$. Moreover, also since $x_n \leq 3$, we have that $4 - x_n \geq 1 > \frac{1}{3}$, so $x_{n+1} = \frac{1}{4-x_n} < 3$. So $0 \leq x_{n+1} \leq 3$. In particular, the sequence (x_n) is bounded below by 0.

Now we claim that (x_n) is decreasing; that is, we claim that $x_n \geq x + n + 1$ for all n. For the base case, we have $x_1 = 3 \geq 1 = x_2$. Now suppose that $x_n \geq x_{n+1}$. Then $4 - x_n \leq 4 - x_{n+1}$. Since by the first part both numbers are positive, it follows that

$$x_{n+1} = \frac{1}{4 - x_n} \geq \frac{1}{4 - x_{n+1}} = x_{n+2}$$

and thus by induction the sequence is decreasing as desired. As (x_n) is bounded below and decreasing, $\lim x_n = x$ exists.

(b) The sequences (x_n) and (x_{n+1}) are the same except for being shifted by an index. If one converges the other converges as well to the same limit.

(c) Taking the limit of both sides using the algebraic limit theorem, we obtain $x = \frac{1}{4-x}$, or $x(4-x) = 1$. We rearrange to $0 = x^2 - 4x + 1$. From the quadratic formula, the solutions to this equation are $2 \pm \sqrt{3}$. Since we are plainly looking for a number less than $x_2 = 1$, we conclude that $\lim x_n = 2 - \sqrt{3}$.

2.4.6

(a) Observe that $(x-y)^2 = x^2 - 2xy + y^2 \geq 0$, so $x^2 + y^2 \geq 2xy$. This implies that

$$(x+y)^2 = x^2 + 2xy + y^2 \geq 4xy.$$

Taking the square root of both sides of this equation shows that $x + y \geq 2\sqrt{xy}$, implying that indeed $\frac{x+y}{2} \geq \sqrt{xy}$.

(b) We have $x_{n+1} = \sqrt{x_n y_n}$ and $y_{n+1} = \frac{x_n + y_n}{2}$, and we start with $0 \leq x_1 \leq y_1$. We claim that for any n, $x_n \leq x_{n+1} \leq y_{n+1} \leq y_n$. For the base case, let $n = 1$. Then $x_2 = \sqrt{x_1 y_1} \geq x_1$ since $x_1 \leq y_1$. Likewise $y_2 = \frac{x_1 + y_1}{2} \leq y_1$. Finally, $x_2 \leq y_2$ by part (a). We conclude that $x_1 \leq x_2 \leq y_2 \leq y_1$. The inductive step follows by identical logic.

Now, we have (x_n) increasing and bounded above by y_1 (or any y_n) and y_n decreasing and bounded below by 0 (or any x_n). We conclude both sequences converge. Thus we may use the Algebraic Limit Theorems and Exercise 2.3.1 to conclude that if $\lim x_n = x$ and $\lim y_n = y$, then $x = \sqrt{xy}$ and $y = \frac{x+y}{2}$. Rearranging either equation gives $x = y$.

2.4.8

(b) For the series $\Sigma_{n=1}^{\infty} \frac{1}{n(n+1)}$, observe that

$$\frac{1}{n(n-1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore the partial sums of the series are of the form

$$\begin{aligned} s_m &= \frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{m(m+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m} - \frac{1}{m+1}\right) \\ &= 1 - \frac{1}{m} \end{aligned}$$

We see that $\Sigma_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

(c) For the series $\Sigma_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$, observe that

$$\log\left(\frac{n+1}{n}\right) = \log(n+1) - \log(n).$$

Therefore the partial sums of the series are of the form

$$\begin{aligned}
s_m &= \log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \cdots + \log\left(\frac{m+1}{m}\right) \\
&= (\log(2) - \log(1)) + (\log(3) - \log(2)) + \cdots + (\log(m+1) - \log(m)) \\
&= \log(m) - \log(1) \\
&= \log(m) - 0 \\
&= \log(m)
\end{aligned}$$

Since $\log(m)$ grows unboundedly as m grows, the series diverges (more precisely, "diverges to infinity."

2 Other Problems

Problem 4

(a) Let (a_n) be a sequence with $a_n \neq 0$ such that $\left|\frac{a_{n+1}}{a_n}\right| = L < 1$. Choose a real number k with $L < k < 1$. Then if $\epsilon = k - L$, there is some N such that $n \geq N$ implies that $\left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| < \epsilon$, implying that $\left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon = k$. In particular for $n \geq N$, we have $\frac{|a_{n+1}|}{|a_n|} < k$, or in other words $|a_{n+1}| < k|a_n|$. Applying this relationship inductively we see that $0 \leq |a_{n+N}| < k^n|a_N|$. Since $k^n \to 0$, we have that $k^n|a_N| \to 0$. Hence by the squeeze theorem $|a_{n+N}| \to 0$. Reindexing this becomes $|a_n| \to 0$. But in general if $|b| \to 0$ then $b \to 0$ (exercise!) so we see that $a_n \to 0$.

(b) Let $b_n = \frac{a^n}{n^p}$ for $|a| < 1$ and $p > 0$. Then we have

$$\left|\frac{b_{n+1}}{b_n}\right| = \frac{|a| \cdot n^p}{(n+1)^p}$$

and the limit of this term as $n \to \infty$ is $|a| < 1$. So $\lim b_n = 0$ by part (a).

(c) Let $c_n = \frac{a^n}{n!}$. Then we have

$$\left|\frac{c_{n+1}}{c_n}\right| = \frac{|a|}{n}$$

and the limit of this term as $n \to \infty$ is $0 < 1$. So $\lim c_n = 0$ by part (a).