Homework 4 Solutions

September 28, 2023

Other Problems

Section 1.5

Problem 1.5.1

Assume B is countable. Then there is a bijection $f : \mathbb{N} \to B$. Let A be an infinite subset of B. Then set $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. Continue inductively, defining n_i to be $\min\{n \in \mathbb{N} : f(n) \in A, n_i > n_{i-1}\}$. Then $g : \mathbb{N} \to A$ defined by $g(i) = f(n_i)$ is a map from$\mathbb{N}$ to A which is injective because f is injective and each of the n_i are distinct, and onto because f is onto, hence a bijection.

Problem 1.5.9

(a) We see that $\sqrt{2}$ is a root of $x^2-2=0$ and that $\sqrt[3]{2}$ is a root of $x^3-2=0$. If $x=\sqrt{2}+\sqrt{3}$, we notice that $x^2 = 5+2\sqrt{6}$, so $(x^2-5)^2 = 24$. Expanding this out we see that $x^4 - 10x^2 - 49 = 0$. So $\sqrt{2}+\sqrt{3}$ is a root of this polynomial.

(b) For $n \in \mathbb{N}$, let P_n be the set of polynomials with integer coefficients having degree n. Notice that P_n is equivalent to an ordered tuple of integers $(a_n, \dots, a_0)$, or an element of the product $\mathbb{Z}^n$. Since $\mathbb{Z}$ is countable, we may apply the usual counting-along-diagonals argument n times to show that the set of all such tuples of integers is countable, so the set P_n is countable. Now we let A_n be the set of all roots of polynomials in P_n. The total number of roots of a polynomial of degree n is finite, so A_n is a union of a countable number (one for each element of P_n) of finite sets. By Theorem 1.5.8, A_n itself is countable.

(c) The set of algebraic numbers is $\cup_{n=1}^{\infty} A_n$, which is a countable union of countable sets, hence countable. We conclude that the set of transcendental numbers must be uncountable.

Section 2.2

Problem 2.2.2

(a) Let $\epsilon > 0$, and choose N such that $N < \frac{25\epsilon}{6}$. Then for $n \geq N$, we have

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{10n+2-(10n+8)}{5(5n+4}\right| = \frac{6}{25n+20} < \frac{6}{25n} < \epsilon.$$

So $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) Let $\epsilon > 0$. Choose N such that $\frac{\epsilon}{2} < N$. Then for $n \geq N$, we have

$$\left|\frac{2n^2}{n^3+3} - 0\right| = \left|\frac{2n^2}{n^3+3}\right| < \left|\frac{2n^2}{n^3}\right| = 2 \cdot \frac{1}{n} < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

So $\lim \frac{2n^2}{n^3+3} = 0$.

(c) Recall that $|\sin x| \leq 1$ for all real x. Given $\epsilon > 0$, let N be a natural number such that $N > \frac{1}{\epsilon^3}$. Then for $n \geq N$, we have

$$\left|\frac{\sin(n^2)}{\sqrt[3]{n}} - 0\right| = \frac{|\sin(n^2)|}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \epsilon.$$

So $\lim \frac{\sin(n^2)}{\sqrt[3]{n}} = 0$.

Section 2.3

Problem 2.3.2

Let $x_n \to 2$.

(a) Given $\epsilon > 0$, choose N such that $n \geq N$ implies that $|x_n - 2| < \frac{3\epsilon}{2}$. Then for $n \geq N$,

$$\left|\frac{2x_n - 1}{3} - 1\right| = \left|\frac{2x_n - 4}{3}\right| = \frac{2|x_n - 2|}{3} < \frac{2}{3} \cdot \frac{3\epsilon}{2} = \epsilon.$$

So $\lim \frac{2x_n-1}{3} = 1$.

(b) First observe that

$$\begin{aligned}\left|\frac{1}{x_n} - \frac{1}{2}\right| &= \left|\frac{2 - x_n}{2x_n}\right| \\ &= \frac{|2 - x_n|}{2|x_n|}\end{aligned}$$

Now choose N_1 such that $n \geq N_1$ implies that $|x_n - 2| < 1$, which is to say $-1 < x_n - 2 < 1$, so that $1 < x_n < 3$, and in particular $x_n > 1$. Then given $\epsilon > 0$, choose N_2 such that $n \geq N_2$ implies that $|2 - x_n| < 2\epsilon$. Then $n \geq N = \max\{N_1, N_2\}$ implies that

$$\begin{aligned}\left|\frac{1}{x_n} - \frac{1}{2}\right| &= \left|\frac{2 - x_n}{2x_n}\right| \\ &= \frac{|2 - x_n|}{2|x_n|} \\ &< \frac{2\epsilon}{2(1)} \\ &= \epsilon.\end{aligned}$$

We see that $\frac{1}{x_n} \to \frac{1}{2}$ as desired.

Problem 2.3.4

(a) Since $a_n \to 0$, the first three algebraic limit theorems imply that $\lim(1+2a_n) = 1+2(0) = 0$ and $\lim(1+3a_n - 4a_n^2) = 1+3(0)-4(0)^2 = 1$. Since in particular the second number is nonzero, we now have that

$$\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) = \frac{1+0}{1+0} = 1.$$

(b) We observe that

$$\frac{(a_n+2)^2-4}{a_n} = \frac{a_n^2+4a_n+4-4}{a_n} = \frac{a_n^2+4a_n}{a_n} = a_n+4.$$

The algebraic limit theorems imply that $\lim(a_n+4) = 0+4 = 4$, so

$$\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) = 4.$$

(c) We observe that

$$\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5} = \frac{2+3a_n}{1+5a_n}.$$

The algebraic limit theorems imply that the limit of the righthand term is 2 as $a_n \to 0$. Ergo

$$\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) = 2.$$

Problem 2.3.8

(a) Let $p(x) = c_m x^m + \cdots + c_1 x + c_0$ be a polynomial and let $x_n \to x$. Then the algebraic limit theorems imply that $p(x_n) = c_m x_n^m + \cdots + c_1 x_n + c_0 \to c_m x^m + \ldots c_1 x + c_0 = p(x)$.

(b) Consider the function $f : [0,1] \to \{0,1\}$ defined by

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Consider the sequence $x_n = \frac{1}{n}$, which converges to $x = 0$. Then $f(x_n) = 1$ but $f(x) = 0$, so $f(x_n)$ does not converge to $f(x)$.

Problem 4

We begin by proving a quick lemma, which will be helpful for Problems 4 and 5.

Lemma 1. *A function $f : A \to B$ is a bijection if and only if there is a function $g : B \to A$ such that $g(f(a)) = a$ for all $a \in A$ and $f(g(b)) = b$ for all $b \in B$. The function g is called the inverse of A.*

Proof. Suppose $f : A \to B$ is a bijection. Then let $g : B \to A$ be the function defined as follows: for any $b \in B$, there exists $a \in A$ such that $f(a) = b$ by surjectivity of f. We let $g(b) = a$. This is well-defined since if $f(a_1) = f(a_2) = b$, then $a_1 = a_2$ by injectivity, so $g(b) = a_1 = a_2$ is a unique element. Then by construction $g(f(a)) = a$. Moreover $f(g(b)) = f(a)$ where a is an element such that $f(a) = b$, so $f(g(b)) = b$.

Now suppose $f : A \to B$ has an inverse g. Then if $f(a_1) = f(a_2)$, we apply g to see that $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$, so f is injective. Moreover, if $b \in B$, we have $f(g(b)) = b$, so b is $f(a)$ for $a = g(b)$. Ergo f is surjective. So f is a bijection. □

Now we look at our prospective equivalence relation. Clearly $f : A \to A$ the identity function given by $f(a) = a$ is a bijection, so $A \sim A$ and this relationship is reflexive. Moreover, suppose $A \sim B$. Then there is a bijection $f : A \to B$, which by the lemma has some inverse function $g : B \to A$. As g also has an inverse, namely f, we have that g is also a bijection. So, $B \sim A$, and $\sim$ is symmetric. Finally, suppose $A \sim B$ and $B \sim C$. Then we have $f : A \to B$ a bijection and $g : B \to C$ a bijection. Consider the function $h = g \circ f : A \to C$ defined by $h(a) = g(f(a))$. First, we claim h is surjective. For if $c \in C$, by surjectivity of g there is some b such that $g(b) = c$, and by surjectivity of f there is some a such that $f(a) = b$. So $h(a) = g(f(a)) = g(b) = c$. As $c \in C$ was arbitrary, h is surjective. Furthermore we claim h is injective. For suppose $h(a_1) = h(a_2)$. Then $g(f(a_1)) = g(f(a_2))$, so by injectivity of g, we have $f(a_1) = f(a_2)$. Then by injectivity of f we have $a_1 = a_2$. So $h(a_1) = h(a_2)$ implies that $a_1 = a_2$, hence h is injective. So h is a bijection and $A \sim C$, so $\sim$ is transitive. Ergo $\sim$ is an equivalence relation.

Problem 5

(a) Consider the map $f : [0, 1) \to [0, 1]$ defined by

$$f(x) = \begin{cases} f(\frac{1}{2^{n-1}}) = \frac{1}{2^n} \\ f(x) = x & x \neq \frac{1}{2^{n-1}} \end{cases}$$

This has inverse $g : [0, 1] \to [0, 1)$ given by

$$f(x) = \begin{cases} f(\frac{1}{2^n}) = \frac{1}{2^{n-1}} \\ f(x) = x & x \neq \frac{1}{2^n} \end{cases}$$

hence is a bijection.

(b) From class, $f(x) = \frac{x}{x^2+1}$ is a bijection from $(-1, 1)$ to $\mathbb{R}$. We also have the map $g : (a, b) \to (-1, 1)$ which is defined by $g(x) = \frac{2(x-a)}{b-a} - 1$, which has inverse $h : (-1, 1) \to (a, b)$ given by $h(x) = \frac{(b-a)(x+1)}{2}$. Then $f \circ g$, the function given by mapping x to $f(g(x))$, is the desired bijection.

(c) Let S be the set of sequences (a_n) such that $a_n \in \{0, 1\}$ and $P(\mathbb{N})$ be the set of subsets of $\mathbb{N}$. Consider the map $f : S \to P(\mathbb{N})$ which sends (a_n) to the subset $A \subset \mathbb{N}$ such that $n \in A$ if an only if $a_n = 0$. This has a clear inverse by reversing the operation, hence is a bijection.

Problem 6

Let r and s be positive rationals. We recall that they correspond to the cuts $r^* = \{p : p < r\} \subseteq \mathbb{Q}$ and $s^* = \{q : q < s\} \subseteq \mathbb{Q}$. The product of these these two cuts is

$$r^*s^* = \{t : t < pq \text{ for some } p \in r^*, q \in s^* \text{ with } p > 0, q > 0\}$$

Our goal is to show that this is equal to $(rs)^* = \{u : u < rs\} \subseteq \mathbb{Q}$. First, let $t \in r^*s^*$. Then there exist $p \in r^*$, $q \in s^*$ with p and q positive such that $t < pq$. But in particular $p < r$ and $q < s$, by definition of r^* and s^*, so in fact $t < pq < rs$, and $t \in (rs)^*$. So, $r^*s^* \subseteq (rs)^*$.

Now, let $u \in (rs)^*$. If $u \leq 0$, then certainly $u \in r^*s^*$, since it is less than the product of any two positive numbers in r^* and s^*. Now, let $u > 0$. Then by assumption $u < rs$. Let

$$\epsilon = \min\left\{\frac{rs-u}{2r}, \frac{rs-u}{2s}, \frac{r}{2}, \frac{s}{2}\right\}.$$

Then we observe that $0 < r - \epsilon < r$ and $0 < s - \epsilon < s$, so that $r - \epsilon \in r^*$ and $s - \epsilon \in s^*$. We have that

$$\begin{aligned}
(r-\epsilon)(s-\epsilon) &= rs - r\epsilon - s\epsilon + \epsilon^2 \\
&> rs - r\epsilon - s\epsilon \\
&\geq rs - \frac{rs-u}{2} - \frac{rs-u}{2} \\
&= rs - (rs - u) \\
&= u.
\end{aligned}$$

Here the third line follows because $r\epsilon \leq r\left(\frac{rs-u}{2r}\right) = \frac{rs-u}{2}$. We conclude that u is less than a product of positive elements in r^* and s^*, hence an element of r^*s^*. So $(rs)^* \subseteq r^*s^*$. Having shown both inclusions we conclude that $(rs)^* = r^*s^*$.