Homework 3 Solutions

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Section 1.3

1.3.5

Let A be nonempty and bounded above, and let $c \in \mathbb{R}$. Set $cA = \{ca : a \in A\}$.

(a) We claim that if $c \geq 0$, then $\sup cA = c \sup A$. First, the statement is obviously true if $c = 0$ since then $cA = \{0\}$ and $c \sup A = 0$. Now let $c > 0$. Let $s = \sup A$. Then $a \leq s$ for all $a \in A$, so $ca \leq cs$ for all $ca \in cA$, so cs is an upper bound for cA. Suppose u is an arbitrary upper bound for cA. Then $ca \leq u$ for all $ca \in cA$, implying that $a \leq \frac{u}{c}$ for all $a \in A$. Ergo $\frac{u}{c}$ is an upper bound for A and in particular $s \leq \frac{u}{c}$. So $cs \leq u$. Hence cs is an upper bound for cA which is less than or equal to any upper bound for cA, and therefore $cs = \sup(cA)$. So $c \sup A = \sup cA$.

(b) We conjecture that if A is nonempty and bounded below and $c < 0$, $c \sup A = \inf cA$. See the final problem for the case that $c = -1$.

1.3.6

Given A and B subsets of the real line, we set $A + B = \{a + b : a \in A, b \in B\}$. Let A and B be bounded above with $s = \sup A$ and $t = \sup B$.

(a) First we observe that since $a \leq s$ for any $a \in A$ and $b \leq t$ for any $b \in B$, for any $a + b$ in $A + B$ we have $a + b \leq s + t$, so $s + t$ is an upper bound for $A + B$.

(b) Now let u be any upper bound for $A + B$. Fix $a \in A$. We see that $a + b \leq u$ for all $b \in B$, implying that $b \leq u - a$ for all $b \in B$. Hence $u - a$ is an upper bound for B, and in particular $t \leq u - a$ since t is less than or equal to any upper bound for B.

(c) Rearranging $t \leq u - a$ for all $a \in A$, we see that $a \leq u - t$ for all $a \in A$. So $u - t$ is an upper bound for A and in particular $s \leq u - t$. Ergo $s + t \leq u$. Since $s + t$ is an upper bound for $A + B$ which is less than or equal to any upper bound for $A + B$, it must be the supremum of $A + B$. So $\sup(A + B) = \sup A + \sup B$.

(d) Alternately, we can use the characterization of the supremum given in Lemma 1.3.8, starting from the fact that $s+t$ is an upper bound for $A+B$ which we showed in part (a). Let $\epsilon > 0$. Since $s = \sup A$, there is some $a \in A$ such that $a > s - \frac{\epsilon}{2}$. Similarly since $t = \sup B$, there is some $b \in B$ such that $b > t - \frac{\epsilon}{2}$. Then we observe that $a + b > s - \frac{\epsilon}{2} + b > t - \frac{\epsilon}{2} = (s+t) - \epsilon$. Since ϵ was arbitrary, we conclude that for all $\epsilon > 0$ there is an element of $A+B$ which is greater than $(s+t) - \epsilon$, and therefore that $s + t = \sup(A+B)$.

1.3.8

The suprema and infima of the sets are as follows.

(a) $\left\{\frac{m}{n} : m, n \in \mathbb{N}, m < n\right\}$. The infimum is zero and the supremum is one, by noticing that $0 < \frac{m}{n} < 1$ for all elements of the set and that $\frac{1}{n}$ can be made arbitrarily close to 0 and $\frac{n-1}{n}$ can be made arbitrarily close to 1.

(b) $\left\{\frac{(-1)^m}{n} : m, n \in \mathbb{N}\right\}$. The infimum, which is also a minimum, is -1 and the supremum, which is also a maximum, is 1.

(c) $\left\{\frac{n}{3n+1} : n \in \mathbb{N}\right\}$. This is an increasing sequence; the infimum, which is also a minimum, is $\frac{1}{4}$ and the supremum is $\frac{1}{3}$.

(d) $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$. This is the same set as part (a)!

Section 1.4

1.4.1

(b) Say $a \in \mathbb{Q}$ and $t \in I$, which is to say a is rational and i is irrational. Suppose that $a + t \in \mathbb{Q}$. Then since the sum of two rationals is a rational and the additive inverse of a rational is a rational, we may add $-a$ to conclude that $a + t + -a = t$ is a rational as well, which is untrue. Similarly, if $a \neq 0$, then if at is rational, since the multiplicative inverse of a nonzero rational is a rational and the product of two rationals is a rational, $a^{-1}at = t$ is as well. We conclude that $a + t$ is irrational and at is irrational as long as $a \neq 0$.

(c) Given two irrational numbers s and t, their sum and product could be either rational or irrational. For example, $\sqrt{2} + -\sqrt{2} = 0 \in \mathbb{Q}$, but $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ is certainly not in $\mathbb{Q}$, for it it were, then its quotient by the rational number 2 would be as well. Similarly, $\sqrt{2}(\sqrt{2}) = 2$ is rational, but $\sqrt{2}\sqrt{3} = \sqrt{6}$ is not, by Homework 1.

1.4.5

We claim that if a and b are real numbers with $a < b$, there is an irrational number t such that $a < t < b$. Consider the real numbers $a - \sqrt{2} < b - \sqrt{2}$. By the density of the rationals in the reals, there is a rational r such that $a - \sqrt{2} < r < b - \sqrt{2}$. Adding $\sqrt{2}$ to all three terms we see that $a < r + \sqrt{2} < b$. Now by Exercise 1.4.1, the sum of a rational number and an irrational number is always irrational, so in particular $r + \sqrt{2}$ is irrational. Therefore we are done.

Other Problems

Problem 5

(a) (iv) Let $a, b \in F$. We recall that part (iii) of this proposition showed us that $(-a)(b) = -ab$. We apply this to show that $(-a)(-b) = -(a(-b)) = -((-b)(a)) = -(-ba) = -(-ab)$, where we have used axiom (A2) twice to commute terms in our product. This tells us that $(-a)(-b) + (-ab) = 0$. But we may add ab to both sides of this equation to obtain $((-a)(-b) + (-ab)) + ab = ab$. Using (A1) we rearrange to $(-a)(-b) + (-ab + ab) = ab$. Using (A4) we may add the terms in parentheses to obtain $(-a)(-b) + 0 = ab$. Using (A3) this reduces to $(-a)(-b) = ab$, as desired.

(v) Let $ac = bc$, with $c \neq 0$. Then by (A4), c has a multiplicative inverse c^{-1}. We multiply by it on both sides to obtain $(ac)c^{-1} = (bc)c^{-1}$. Using (M1) we rearrange to obtain $a(cc^{-1}) = b(cc^{-1})$. Using (M4) this reduces to $a(1) = b(1)$ and using (M3) this becomes $a = b$, as desired.

(b) (v) Recall that part (iv) tells us that all squares in an ordered field are nonnegative. But $1 = (1)(1)$ is a square, hence nonnegative, and $0 \neq 1$, so we have that $0 < 1$.

(vi) We have $0 < a$. Suppose that $a^{-1} < 0$. Then by invariance under multiplication by a positive, we have $a^{-1}a < a(0)$, or $1 < 0$. This is false, by (v). Ergo, $0 < a^{-1}$.

(vii) Say that $0 < a < b$. We know by (vi) that both a^{-1} and b^{-1} are positive. Now, multiply both sides by $a^{-1}b^{-1}$. Then we have $a(a^{-1}b^{-1}) < b(a^{-1}b^{-1})$. Applying (M1) on the left and (M2) on the right we obtain $(aa^{-1})b^{-1} < b(b^{-1}a^{-1})$. Now applying (M4) on the left and (M1) on the right we obtain $1(b^{-1}) < (bb^{-1})a^{-1}$. Now applying (M3) on the left and (M4) on the right, we obtain $b^{-1} < 1(a^{-1})$. A final application of (M3) gives $b^{-} < a^{-1}$. So $0 < b^{-} < a^{-1}$ as desired.

Problem 6

(a) Suppose we can make $\mathbb{Z}/p\mathbb{Z}$ into an ordered field. In $\mathbb{Z}/p\mathbb{Z}$, the additive identity element is $0 = [0]$ and the multiplicative identity element is $1 = [1]$. We recall that in any ordered field the additive identity is less than the multiplicative identity, that is, $[0] < [1]$. But then adding $[1]$ to both sides we have $[1] < [2]$, $[2] < [3]$, and so on. In particular using transitivity $[1] < [2] < \cdots < [p-1] < [p] = [0]$. But then $[1] < [0]$, which violates trichotomy. So, there is no such order on $\mathbb{Z}/p\mathbb{Z}$.

(b) (i) It is easy to see that the multiplicative identity in $\mathbb{C}$ is the element $1 = 1 + 0i$, since $(1+0i)(a+bi) = (1(a) - b(0)) + (0(a) + 1(b))i = a + bi$, and likewise for $(a+bi)(1+0i)$. Now, let $a + bi$ be nonzero in $\mathbb{C}$; since the additive identity is $0 + 0i$, this in particular means that at least one of a and b is nonzero. We claim that the multiplicative inverse of $a + bi$ is $\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$. Note that if $a + bi \neq 0$, then $a^2 + b^2 > 0$, so this is a valid element of $\mathbb{C}$. We check it is actually the multiplicative inverse:

$$\begin{aligned}
(a+bi)\left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i\right) &= \left(a\left(\frac{a}{a^2+b^2}\right) - b\left(\frac{-b}{a^2+b^2}\right)\right) + \left(a\left(\frac{-b}{a^2+b^2}\right) + b\left(\frac{a}{a^2+b^2}\right)\right)i \\
&= \frac{a^2+b^2}{a^2+b^2} + \frac{-ba+ab}{a^2+b^2}i \\
&= 1 + 0i \\
&= 1
\end{aligned}$$

The other product is similar, or we may use the fact that multiplication commutes.

(ii) We recall that in an ordered field all squares are positive. Ergo, if $\mathbb{C}$ can be made into an ordered field, then $0 < -1$, since $i^2 = -1$. But $0 < 1$ as well, and if an element in an ordered field is positive, then its additive inverse is negative. So no such ordering exists.

Problem 7

Let S be nonempty and bounded below, say by ℓ. Then for any $s \in S$, we have $\ell \leq s$, implying that $-s \leq -\ell$. So ℓ is an upper bound for $-S$. Since $-S$ is bounded above it has a supremum by the Axiom of Completeness, call it a. We claim that $-a$ is the infimum of S. First, since a is an upper bound of $-S$, we have $-s \leq a$ for all $-s \in -S$, so we see that $-a \leq s$ for all $s \in S$, hence $-a$ is a lower bound for S. Furthermore suppose that ℓ is an arbitrary lower bound for S, then as we argued above $-\ell$ is an upper bound for $-S$, so $a \leq -\ell$, implying that $\ell \leq -a$. Hence a is greater than or equal to any lower bound for S. Ergo a is the infimum of S. We conclude that bounded below nonempty subsets of the real numbers have infima.