MATH 311H: Homework 3

Due: September 25 at 5 pm

1. Upcoming office hours are Monday September 18 and Thursday September 21 10-11 am in LSH B-102D.

2. A reminder that the thirty minute warm-up quiz is Thursday September 28, and will cover up to the end of Chapter 1 (which is to say, until midway through lecture on Thursday September 21). There will be three questions in total.

3. Read Section 8.6 (a construction of $\mathbb{R}$) and 2.1-2 in Abbott.

4. Do Abbott exercises 1.3.5*, 1.3.6, 1.3.8, 1.4.1(b),(c)*, 1.4.5

5. (a) Prove that for a field F, the following statements hold

   (iv) $(-a)(-b) = ab$ for all $a, b \in F$*

   (v) If $ac = bc$ and $c \neq 0$, then $a = b$

   (b) Prove that for an ordered field F, the following statements are true.

   (v) $0 < 1$ [Note that we will require $0 \neq 1$, so that our field has at least two elements.]

   (vi)* If $0 < a$, then $0 < a^{-1}$

   (vii) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$

   The numbering here is drawn from the statements of the propositions containing these claims in class; in each case you may if you like use previous statements from the proposition.

6. (a) Given a prime p, let $\mathbb{Z}/p\mathbb{Z}$ be the field defined on Homework 2. Prove that $\mathbb{Z}/p\mathbb{Z}$ cannot be given the structure of an ordered field.*

   (b) Recall that the complex numbers $\mathbb{C}$ are the set of all numbers $a + bi$ such that $a, b \in \mathbb{R}$ and i is a number satisfying $i^2 = -1$, with operations given by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i$$

   (i) It turns out $\mathbb{C}$ is a field. The most interesting axiom to check is (M4); give a proof that it holds. (You do not need to check the others and in particular may assert what the additive and multiplicative identity elements are without proof.)

   (ii) Show there is no relation $\leq$ on $\mathbb{C}$ which makes $\mathbb{C}$ into an ordered field.

7. Given a set S in $\mathbb{R}$, let $-S$ be the set $\{-s : s \in S\}$.

   (a) Prove that if S is bounded below, $-S$ is bounded above and $\sup(-S) = -\inf S$.

   (b) Use this to conclude that the Axiom of Completeness implies that every bounded below subset of $\mathbb{R}$ has an infimum.

   [Remark: Abbott Exercise 1.3.3 contains a different proof of this fact.]