MATH 311H: Homework 2

Due: September 18 at 5 pm

1. Upcoming office hours are Monday September 11 and Thursday September 14 10-11 am in LSH B-106. This is our first attempt at using this room reservation; if there are any problems we will relocate to the Livingston Student Center.

2. Read Sections 1.4-6 in Abbott.

3. Do Abbott Exercise 1.2.3*. You don't have to prove the true statements, just provide counterexamples for the false ones.

4. Do Abbott Exercise 1.2.5 part (c). [We did parts (a) and (b) in lecture and your proof should look similar.]

5. Given a function $f : A \to B$ and a subset $C \subseteq A$, the image of C is the set $\{f(c) : c \in C\} \subseteq B$.

   (a) Given $C, D \subseteq A$, show that $f(C \cap D) \subseteq f(C) \cap f(D)$.*

   (b) Give an example to show the relationship in part (a) need not be an equality.

6. In class we gave an operation of addition and multiplication on $\mathbb{Q}$ as constructed from a partition of $\mathbb{Z} \times (\mathbb{Z} - \{0\})$. Prove that

   (a) The binary relation on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$ given by $(m, n) \sim (m', n')$ if $mn' = nm'$ is indeed an equivalence relation.*

   (b) The operations of addition and multiplication given by

$$[(m, n)] + [(p, q)] = [(mq + np, nq)]$$

$$[(m, n)] \times [(p, q)] = [(mp, nq)]$$

   are well-defined.*

7. In class we defined a set $\mathbb{Z}/k\mathbb{Z}$ consisting of the equivalence classes of $\mathbb{Z}$ under the relation $a \equiv b \pmod{k}$ if $k|(a - b)$, so that the elements of $\mathbb{Z}/k\mathbb{Z}$ are the equivalence classes $\{[0], \ldots [k - 1]\}$. Consider the operations

$$[a] + [b] = [a + b]$$

$$[a] \times [b] = [ab]$$

   (a) Prove that these two operations are well-defined and satisfy the axioms (A1)-(A4), (M1)-(M3), and (DL) of a field. (That is, they give $\mathbb{Z}/k\mathbb{Z}$ the structure of a commutative unital ring.)

   (b) Give an example to show that the structure defined above need not satisfy the axiom (M4).

   (c) Prove that if p is a prime, then these operations also satisfy (M4); that is, that they give $\mathbb{Z}/p\mathbb{Z}$ the structure of a field. [Hint: one way to do this is to show that for $[a] \neq [0]$, the classes $\{[a(0)], \ldots, [a(p - 1)]\}$ must be distinct.]

   (d) What is the multiplicative inverse of [2] in $\mathbb{Z}/3\mathbb{Z}$? In $\mathbb{Z}/5\mathbb{Z}$?