Homework 1 Solutions

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Section 1.2

Problem 1.2.1

(a) We wish to prove that $\sqrt{3}$ is not a rational number. For the sake of contradiction, suppose that

$$\sqrt{3} = \frac{p}{q}$$

where p and q are positive coprime integers. Then we have

$$3 = \frac{p^2}{q^2}$$

implying that $3q^2 = p^2$. This implies that $3|p^2$, and therefore $3|p$. Let $p = 3k$ for some natural number k. Then $3q^2 = 9k^2$, so in fact $q^2 = 3k^2$, implying that $3|q^2$ and therefore $3|q$. But we assumed that p and q are coprime, so this is a contradiction, and $\sqrt{3} \notin \mathbb{Q}$.

This argument only has to be refined slightly to show that $\sqrt{6}$ is irrational. If we assume that

$$\sqrt{6} = \frac{p}{q}$$

then we have $6q^2 = p^2$. This implies that p^2 is divisible by 6, so in particular by 3, and so is p. Then $p = 3k$ for some natural number k, so we have $6q^2 = 9k^2$, implying that $2q^2 = 3k^2$, so q^2 is divisible by 3, and therefore so is q. Since p and q were coprime, this is a contradiction, and $\sqrt{6} \notin \mathbb{Q}$.

(b) If we try to run this argument by setting $\sqrt{4} = \frac{p}{q}$, we get that $4q^2 = p^2$, which implies that $4|p^2$. This does not necessarily imply that $4|p$. Instead, it implies that $2|p$. But if $p = 2k$, we get $4q^2 = 4k^2$, so $q^2 = k^2$, which is not a contradiction.

Problem 1.2.2

Suppose that $2^{\frac{p}{q}} = 3$, where p and q are coprime integers such that q is positive. Then we raise both sides of the equation by a power of q, obtaining $2^p = 3^q$. Then the number 3^q is an integer divisible by 3 (since q was assumed positive), which is not true of any power of q. Hence this is impossible.

Problem 1.2.12

We begin with $y_1 = 6$ and $y_{n+1} = \frac{2y_n - 6}{3}$.

(a) We claim that $y_n > -6$ for all $n \in \mathbb{N}$. Let this statement be P_n. For the base case, clearly $y_1 = 6 > -6$, so P_1 is true. Now suppose that P_n is true, so that $y_n > -6$. Then we have $2y_n > -12$, implying that $2y_n - 6 > -18$. Dividing both sides by 3 we obtain that

$$y_{n+1} = \frac{2y_n - 6}{3} > -6$$

so P_{n+1} follows from P_n. Hence the statement is true for all n by induction.

(b) We claim that $y_n > y_{n+1}$ for all $n \in \mathbb{N}$. Let this statement be P_n. For a base case, $y_1 = 6$ and $y_2 = \frac{2(6)-6}{3} = 2$, so we see that $y_1 > y_2$. For the inductive step, suppose that P_n is true. Then we have $y_n > y_{n+1}$, implying that $2y_n > 2y_{n+1}$, and furthermore that $2y_n - 6 > 2y_{n+1} - 6$. Dividing both sides by 3 we obtain

$$y_{n+1} = \frac{2y_n - 6}{3} > \frac{2y_{n+1} - 6}{3} = y_{n+2}.$$

Ergo P_{n+1} follows from P_n, so we conclude that the statement is true for all n.

Other Problems

Problem 5

We claim that 7 divides $(11)^n - 4^n$ for all $n \in \mathbb{N}$. Let this statement be P_n. We see that for $n = 1$, $11 - 4 = 7$, so the statement is true for the base case P_1. For the inductive step, assume that P_n is true, that is, that $(11)^n - 4^n = 7k$ for some positive integer k. Then we have

$$\begin{aligned}(11)^{n+1} - 4^{n+1} &= 11(11^n) - 4(4^n) \\ &= (7+4)(11^n) - 4(4^n) \\ &= 7(11^n) + 4(11^n - 4^n) \\ &= 7(11^n) + 4(7k) \\ &= 7(11^n + 4k)\end{aligned}$$

which shows that $(11)^{n+1} - 4^{n+1}$ is divisible by 7. Ergo, P_{n+1} follows from P_n, so the statement is true for all n by induction.

Problem 6

We wish to prove that $(1+x)^n > 1 + nx$ for $n > 1$. We proceed by induction. The base case P_2 is $n = 2$, which is true since $(1+x)^2 = 1 + 2x + x^2 > 1 + 2x$. Now assume that the nth case P_n is true, and consider $(1+x)^{n+1} = (1+x)(1+x)^n$. Because $1 + x > 0$ and $(1+x)^n > 1 + nx$ by assumption, we see that $(1+x)^{n+1} > (1+x)(1+nx) = 1 + (n+1)x + nx^2 > 1 + (n+1)x$, where the last step follows because nx^2 is a positive number. Therefore if P_n is true, P_{n+1} is also true, and the claim follows.

Problem 7

(a) The inductive step is flawed when $n = 2$. In particular, suppose we have a set of two horses $\{x_1, x_2\}$. Then the set $A_1 = \{x_1\}$ and $A_2 = \{x_2\}$ have no overlap. Therefore we cannot conclude that any two horses have the same color, and thus cannot induct to larger sets of horses (even though the inductive step is valid for higher n).

(b) One correct answer is that this is not a good proof because we have failed to give a definition of "interesting," and indeed, seem to have changed whatever definition we were using midway through the argument. This illustrates the importance of defining mathematical terms carefully instead of relying upon their colloquial meanings.