Homework 13 Solutions

December 5, 2023

Section 6.2

Problem 6.2.1

We have $f_n(x) = \frac{nx}{1+nx^2}$.

(a) The pointwise limit of (f_n) on $(0, \infty)$ is $f(x) = \frac{1}{x}$.

(b) The convergence on $(0, \infty)$ is not uniform. Suppose that for $\epsilon > 0$, there exists N such that $n \geq N$ implies that $|f_n(x) - f(x)| < \epsilon$ for all $x \in (0, \infty)$. But also $f_n(0) = 0$ for all n, so it is also true that $n \geq N$ implies that $f_n(x) - f(x)| < \epsilon$ for $n \geq N$. So we have that f_n converges uniformly on $[0, \infty)$ to

$$f(x) = \begin{cases} \frac{1}{x} & x \in (0, \infty) \\ 0 & x = 0 \end{cases}$$

But this is a contradiction, since the uniform limit of continuous functions is continuous. We conclude that in fact the convergence is not uniform on $(0, \infty)$. (You could also argue from the fact that each f_n is bounded and f is not.)

(c) By the same argument the convergence is also not uniform on $(0, 1)$.

(d) The convergence is uniform on $(1, \infty)$. We observe that

$$\begin{aligned} |f(x) - f_n(x)| &= \left| \frac{1}{x} - \frac{nx}{1+nx^2} \right| \\ &= \left| \frac{1 + nx^2 - nx^2}{x(1+nx^2)} \right| \\ &= \frac{1}{x(1+nx^2)} \\ &\leq \frac{1}{1+nx^2} \\ &\leq \frac{1}{1+n} \\ &< \frac{1}{n}. \end{aligned}$$

The third and fourth steps above use the fact that $x > 1$. So given $\epsilon > 0$, if N is such that $\frac{1}{N} < \epsilon$, then $n \geq N$ implies that $|f_(x) - f_n(x)| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$.

Problem 6.2.3

We start by considering $g_n(x) = \frac{x}{1+x^n}$.

(a) We see that the pointwise limit of (g_n) on $[0,\infty)$ is

$$g(x) = \begin{cases} 0 & x \in [0,1) \cup (1,\infty) \\ \frac{1}{2} & x = 1 \end{cases}$$

(b) We see the convergence cannot be uniform on $[0,\infty)$ since the uniform limit of continuous functions is continuous.

(c) Consider $A = [2,\infty)$. On this domain we see that

$$\begin{aligned} |g_n(x) - g(x)| &= \left| \frac{x}{1+x^n} - 0 \right| \\ &= \frac{x}{1+x^n} \\ &< \frac{1}{x^{n-1}} \\ &\leq \frac{1}{2^{n-1}} \end{aligned}$$

So given $\epsilon > 0$, we can choose N such that $\frac{1}{2^{N-1}} < \epsilon$, and then for $n \geq N$ we have $|g_n(x) - g(x)| < \epsilon$. Thus the convergence is uniform on $[2,\infty)$.
We now consider (h_n) defined by

$$h_n(x) = \begin{cases} 1 & x \geq \frac{1}{n} \\ nx & 0 \leq x < \frac{1}{n} \end{cases}$$

(a) We see the pointwise limit is

$$h(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \end{cases}$$

(b) It is still the case that the uniform limit of continuous functions is continuous, so the convergence cannot be uniform on $[0,\infty)$.

(c) Consider $A = [a,\infty)$ where $a > 0$. There exists N such that $\frac{1}{N} < a$. Ergo, for $n \geq N$, we have $|h_n(x) - h(x)| = |1 - 1| = 0$ for all $x \in [a,\infty)$, which is in particular less than any ϵ we care to name. Ergo, the convergence is uniform on this A.

Section 6.3

Problem 6.3.3

(a) Consider $f_n(x) = \frac{x}{1+nx^2}$. Its pointwise limit is $f(x) \equiv 0$. We observe that the functions f_n are everywhere differentiable with derivative

$$f_n'(x) = \frac{1(1+nx^2) - x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

The points where this function has trivial derivative are $x = \pm\sqrt{\frac{1}{n}}$, at which point $f\left(\pm\sqrt{\frac{1}{n}}\right) = \pm\frac{1}{2\sqrt{n}}$. This implies that for all $x \in \mathbb{R}$ we have

$$|f_n(x) - f(x)| = |f_n(x) - 0| \leq \frac{1}{2\sqrt{n}}.$$

So, for any $\epsilon > 0$, we need only choose N such that $\frac{1}{2\sqrt{N}} < \epsilon$ to show that $|f_n(x) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$ and $n \geq N$.

(b) Let us consider the pointwise limit of f_n', call it g. We see that

$$g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

We see that this agrees with $f'(x)$ away from 0. (This means the convergence of (f_n') cannot be uniform on any interval around 0.)

Section 6.4

Problem 6.4.7

Consider $f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}$. We start by observing that since $\left|\frac{\sin(kx)}{k^3}\right| \leq \frac{1}{k^3}$, by the M-test, we have that $f(x)$ converges uniformly on $\mathbb{R}$.

(a) To show f is differentiable, it suffices to show the series of derivatives of the terms converges uniformly (since we already know that f itself converges at at least one point; indeed, at every point). But the series of derivatives is

$$g(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

As $\left|\frac{\cos(kx)}{k^2}\right| < \frac{1}{k^2}$, by the M-test, $g(x)$ converges uniformly. So, by the Differentiable Limit Theorem, $f'(x) = g(x)$.

(b) We have no idea. The question is whether $g(x)$ is differentiable, and the problem is that the series consisting of the derivatives of the terms of g is

$$h(x) = \sum_{k=1}^{\infty} \frac{-\cos(kx)}{k}$$

There is not an obvious way to use the M-test on this series since the harmonic series does not converge. So we do not know anything about the convergence of h, and therefore draw no conclusions from the Differentiable Limit Theorem.

Section 6.5

Problem 6.5.6

Recall that we have

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

on the interval $(-1, 1)$ and that power series may be differentiated term-by-term, with the derivative having the same radius of convergence as the original. We see that

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots.$$

on the interval $(-1, 1)$. It follows that

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \dots = \sum_{n=1}^{\infty} nx^n.$$

on the interval $(-1, 1)$. Now we have that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = \frac{1}{2(\frac{1}{4})} = 2.$$

Differentiating again, we see that on $(-1, 1)$, we have

$$\frac{1(1-x)^2 - x(-2)(1-x)}{(1-x)^4} = \frac{1-x^2}{(1-x)^4} = 1 + 4x + 9x^2 + \dots.$$

We multiply by x to see that on $(-1, 1)$, we have

$$\frac{x(1-x^2)}{(1-x)^4} = x + 4x^2 + 9x^3 + \dots = \sum_{n=1}^{\infty} n^2 x^n.$$

Ergo

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2}(1-\frac{1}{4})}{(1-\frac{1}{2})^4} = \frac{\frac{3}{8}}{\frac{1}{16}} = 6.$$

Other Problems

Problem 5

(a) Clearly $\sum n^2 x^n$ diverges when $x \neq 0$, so the radius of convergence is $R = 0$ and the interval of convergence is $\{0\}$.

(b) Let $\sum a_n x^n = \sum (\frac{2^n}{n^2}) x^n$. We observe that $|\frac{a_{n+1}}{a_n}| = |\frac{2n^2}{(n+1)^2}| \to 2 = \beta$ as $n \to \infty$, so the radius of convergence is $R = \frac{1}{2}$. The endpoints of the interval of convergence are $\pm\frac{1}{2}$; we observe that $f(\frac{1}{2}) = \sum \frac{1}{n^2}$ and $f(-\frac{1}{2}) = \sum \frac{(-1)^n}{n^2}$, both of which converge. Thus the interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$.

(c) Let $\sum a_n x^n = \sum(\frac{2^n}{n!})x^n$. We observe that $|\frac{a_{n+1}}{a_n}| = |\frac{2}{n+1}| \to 0$ as $n \to \infty$. Therefore the radius of convergence is $R = \infty$, and the interval of convergence is $\mathbb{R}$.

(d) Let $\sum a_n x^n = \sum(\frac{3^n}{n \cdot 4^n})x^n$. We observe that $|\frac{a_{n+1}}{a_n}| = |\frac{3n}{4(n+1)}| \to \frac{3}{4} = \beta$ as $n \to \infty$. Therefore the radius of convergence is $R = \frac{4}{3}$. The endpoints of the interval of convergence are $\pm\frac{4}{3}$; we observe that $f(\frac{4}{3}) = \sum \frac{1}{n}$, which diverges, and $f(-\frac{4}{3}) = \sum \frac{(-1)^n}{n}$, which converges. So the interval of convergence is $[-\frac{4}{3}, \frac{4}{3})$.

(e) Notice that a_n is equal to $(\frac{2}{5})^n$ if n is odd and $(\frac{6}{5})^n$ if n is odd. Since the terms of a convergent series must converge to zero as a sequence, we observe that the power series $\sum a_n x^n$ cannot converge for any x with $|x| \geq \frac{5}{6}$, since for such an x the odd index terms do not tend to zero. But for any x with $|x| < \frac{5}{6}$, we see that $|a_n x^n| \leq (\frac{6}{5})^n |x|^n$, and the term on the right converges, so the series itself converges absolutely. We conclude that in fact $R = \frac{5}{6}$ and the interval of convergence is $(-\frac{5}{6}, \frac{5}{6})$.