Homework 12 Solutions

November 20, 2023

Section 5.2

Problem 5.2.2

(a) Possible. Let $f(x) = x^{\frac{1}{3}}$ and $g(x) = x^{\frac{2}{3}}$, such that neither f nor g is differentiable at 0 but $fg(x) = x$ is.

(b) Possible. Let $g(x) \equiv 0$ be the constant zero function and $f(x)$ be any function not differentiable at 0. Then $fg(x) \equiv 0$ is differentiable at 0. [Note that if we added the hypothesis that $g(0)$ and $g'(0)$ are nonzero, this would become impossible – why?]

(c) Impossible. Suppose g and $f + g$ are both differentiable at zero. Then $f = (f + g) - g$ is also differentiable at 0 by the Algebraic Differentiability Theorem.

(d) Possible. Consider

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

which we saw in class is differentiable only at $c = 0$.

0.1 Problem 5.2.9

(a) True; derivatives have the intermediate value property and therefore if the derivative of f is defined on an interval A and not constant, its image $f'(A)$ is a nondegenerate interval (that is, not a single point), and therefore contains irrational values.

(b) False. Consider the example

$$f(x) = \begin{cases} 2x^2 \sin\left(\frac{1}{x}\right) + x & x > 0 \\ x & x \leq 0 \end{cases}$$

whose derivative is

$$f'(x) = \begin{cases} 4x \sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right) + 1 & x > 0 \\ 1 & x \leq 0 \end{cases}$$

We see that $f'(0) = 1 > 0$. However, for any $\epsilon > 0$, choose n a natural number large enough that $x = \frac{1}{2\pi n} < \epsilon$. Then $f'(x) = 0 - 2 + 1 = -1$. So there is no interval around 0 on which $f'(x)$ is nonzero.

(c) True. For suppose not, then we have $f'(0) = a$ and $\lim_{x\to 0} f'(x) = L$ for $L \neq a$. Let $\epsilon = \frac{1}{2}|a - L|$, and choose δ such that $0 < |x - 0| < \delta$ implies $|f'(x) - L| < \epsilon$. So on the interval $(-\delta, \delta)$, the image of f' lies entirely inside the interval $(L - \epsilon, L + \epsilon)$ except that it contains the point a which is outside this interval. This is a contradiction, since derivatives have the intermediate value property, which implies that the image of an interval under the derivative function is an interval.

Section 5.3

Problem 5.3.2

Suppose f is differentiable on an interval A. Then f is also continuous on A. Suppose that f is not one-to-one on A, that is, suppose there is some $x < y$ such that $f(x) = f(y)$. Then by the Mean Value Theorem applied to $[x, y]$, there is some $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{x - y} = \frac{0}{x - y} = 0$$

This implies that $f'(c) = 0$. So, if $f'(c) \neq 0$ on all of A, f must be one-to-one on A.

The converse is not true by considering the example of $f(x) = x^3$, which is one-to-one on $\mathbb{R}$ but has $f'(0) = 0$.

Problem 5.3.3

We have h a differentiable, and therefore also continuous, function on $[0, 3]$ such that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

(a) Consider the function $f(x) = h(x) - x$. We observe that $f(0) = 1 - 0 = 1 > 0$ and $f(3) = 2 - 3 = -1 < 0$. By the Intermediate Value Theorem, we see there is a point d in $(0, 3)$ at which $f(d) = 0$, which means that $h(d) = d$.

(b) By the Mean Value Theorem, there is some $c \in (0, 3)$ with the property that

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{2 - 1}{3} = \frac{1}{3}.$$

(c) By Rolle's Theorem, since $h(1) = h(3)$, there is some $e \in (1, 3)$ with the property that $h'(e) = 0$. Now, recall that derivatives have the intermediate value property; the image of an interval under the derivative is an interval. We have seen that $h'([0, 3])$ contains 0 and $\frac{1}{3}$; ergo, it contains all values between them, including $\frac{1}{4}$. So there is some point x in the domain for which $h'(x) = \frac{1}{4}$.

Other Problems

0.2 Problem 4

Let $y < x$ be two real numbers. Since $f(x) = \cos x$ is continuous and differentiable with derivative $f'(x) = -\sin x$ everywhere, the Mean Value Theorem tells us there is some $c \in (y, x)$ with the property that

$$-\sin c = \frac{\cos x - \cos y}{x - y}$$

But $|-\sin c| \leq 1$. So

$$1 \geq \frac{|\cos x - \cos y|}{|x-y|}$$

implying that $|x-y| \geq |\cos x - \cos y|$.

0.3 Problem 5

(a) To evaluate $\lim_{x\to 0} \frac{1-\cos x}{x^2}$, we observe that the numerator and denominator are both differentiable on a neighborhood of 0 and both evaluate to 0 at 0. So we may apply L'Hospital's Rule to see that

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x}$$

if we can show that the limit on the right exists. However, we see that $\frac{\sin x}{2x}$ still has the property that both the numerator and denominator are differentiable on a neighborhood of 0 and both evaluate to 0 at 0. Ergo we apply L'Hospital's Rule again, seeing that

$$\lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

We conclude that $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

(b) To evaluate $\lim_{x\to 0} \left[\frac{1}{\sin x} - \frac{1}{x}\right]$, we first rewrite the limit as

$$\lim_{x\to 0} \frac{x - \sin x}{x \sin x}$$

We use two successive applications of L'Hospital's Rule, noting at each step that the numerator and denominator are both differentiable on a neighborhood of 0 and evaluate to 0 at 0, to observe that

$$\lim_{x\to 0} \frac{x-\sin x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2-0} = 0.$$

(c) To evaluate $\lim_{x\to 0}(1+2x)^{\frac{1}{x}}$, we in fact start by computing $\lim_{x\to 0} \ln[(1+2x)^{\frac{1}{x}}]$. We observe that

$$\ln[(1+2x)^{\frac{1}{x}}] = \frac{1}{x} \cdot \ln(1+2x) = \frac{\ln(1+2x)}{x}.$$

Since $\ln(1+2x)$ and x are both differentiable on a neighborhood of 1 and both evaluate to 0 at 0, we apply L'Hospital's Rule to see that

$$\lim_{x\to 0} \frac{\ln(1+2x)}{x} = \lim_{x\to 0} \frac{\frac{2}{1+2x}}{1} = 2.$$

Now we finish the problem. Since the function e^x is continuous on $\mathbb{R}$, we have that

$$\begin{aligned}\lim_{x\to 0}(1+2x)^{\frac{1}{x}} &= \lim_{x\to 0} e^{\ln[(1+2x)^{\frac{1}{x}}]} \\ &= e^{\lim_{x\to 0} \ln[(1+2x)^{\frac{1}{x}}]} \\ &= e^2.\end{aligned}$$

Problem 6

(a) We may consider the constant functions.

(b) Recall from Midterm 2 that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is determined by its values on the rational numbers, and therefore by the sequence $(f(r_1), f(r_2), \dots)$ where $(r_1, r_2, r_3, \dots)$ is any enumeration of the rationals. Therefore the set A of functions continuous on all of $\mathbb{R}$ injects into the set of sequences of real numbers.

0.3.1 (c) By applying the bijection between $(0, 1)$ and $\mathbb{R}$ to part (b) we can in fact inject A into the set of sequences of elements of $(0, 1)$. Now we construct an injection from the set of sequences of elements of $(0, 1)$ into $(0, 1)$ itself, and then apply the bijection between $(0, 1)$ and $\mathbb{R}$). For this purpose, let

$$(.x_{11}x_{12}x_{13}\dots, .x_{21}x_{22}x_{23}\dots, .x_{31}x_{32}x_{33}, \dots)$$

be a sequence of real numbers in $(0, 1)$, written as their base ten decimal expansions, none of which terminate in a string of infinite 9's. Then we may construct a real number

$$.x_{11}x_{12}x_{21}x_{13}x_{22}x_{31}\dots$$

by going along the finite diagonals of a grid of all of the digits involved; note that it does not conclude with a string of infinite 9's, because this would require all of the original numbers in the sequence to do so. (Note that it is not immediately obvious this map is a bijection, because of the issue of strings of 9's.) Finally, we include $(0, 1)$ into $\mathbb{R}$ to show A injects into $\mathbb{R}$.

(d) We see that $\mathbb{R}$ injects into A and A injects into $\mathbb{R}$, so A and $\mathbb{R}$ have the same cardinality. Likewise for B.

(e) Notice that the power set $P(\mathbb{R})$ can be regarded as a subset of C via the following method: if $S \subset \mathbb{R}$ is a subset of $\mathbb{R}$, we may consider the function $f : \mathbb{R} \to \mathbb{R}$ by setting $f(x) = 1$ if $x \in \mathbb{R}$ and $f(x) = 0$ otherwise. From Theorem 1.6.2, $P(\mathbb{R})$ has cardinality larger than $\mathbb{R}$. Therefore so does C.

Problem 7

Let $f(x)$ be as in the problem statement. We showed in class this is infinitely differentiable. Here are the appropriate functions:

(a) Let $f_a(x) = f(x - a)$.

(b)Let $g_b(x) = f(b - x)$.

(c) Let $h_{a,b}(x) = f_a(x)g_b(x)$.

(d) Let $j_{a,b}(x) = \frac{f_a(x)}{f_a(x)+g_b(x)}$. Notice that for $x \geq b$, $j_{a,b}(x) = \frac{f_a(x)}{f_a(x)+0} = 1$. Furthermore, we claim that that $f_a(x) + g_b(x) > 0$ on all x: on $x \leq a$, $f_a(x) = 0$ and $g_b(x) > 0$, on $a < x < b$, both $f_a(x) > 0$ and $g_b(x) > 0$, and on $x \geq b$, $f_a(x) > 0$ and $g_b(x) = 0$. So the denominator is always positive and $j_{a,b}$ is infinitely differentiable.