MATH 311H: Homework 12

Due: December 4 at 5 pm

1. Office hours this week are Monday, November 27 2-3 on Zoom at Meeting ID 570 840 4797 with passcode cycle, and Thursday, November 30 10-11 in LSH 102D.

2. Read Sections 5.2-4, 6.1-3 in Abbott.

3. Do exercises 5.2.2, 5.2.9*, 5.3.2, 5.3.3* in Abbott.

4. Prove that $|\cos x - \cos y| \leq |x - y|$ for all x and y in $\mathbb{R}$.

5. Find the following limits if they exist.

   - $\lim_{x \to 0} \frac{1-\cos x}{x^2}$
   - $\lim_{x \to 0} \left[\frac{1}{\sin x} - \frac{1}{x}\right]$
   - $\lim_{x \to 0}(1 + 2x)^{\frac{1}{x}}$

6. Let A be the set of functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous at every point of $\mathbb{R}$ and B be the set of functions $g : \mathbb{R} \to \mathbb{R}$ which are differentiable at every point of $\mathbb{R}$. Let C be the set of all functions $h : \mathbb{R} \to \mathbb{R}$.*

   (a) Prove that there are injections $\mathbb{R} \hookrightarrow A$ and $\mathbb{R} \hookrightarrow B$.

   (b) Prove that A injects into the set S of sequences of real numbers, and therefore that $B \subseteq A$ does as well. Hint: Recall Problem 2 on Midterm 2.

   (c) Show that the set of sequences of elements of $(0, 1)$ injects into $(0, 1)$. Use this to conclude that A injects into $\mathbb{R}$, and therefore B does as well. Hint: Decimal expansions and diagonals.

   (d) Use Exercise 1.5.11 (you do not need to prove it) to conclude that both A and B have the same cardinality of $\mathbb{R}$.

   (e) Now construct an injection from the power set $P(\mathbb{R})$ of $\mathbb{R}$ – that is, the set of subsets of $\mathbb{R}$ – into C and use this to argue that C has larger cardinality than $\mathbb{R}$.

   I believe both Exercise 1.5.11 and Theorem 1.6.2 were previously discussed in recitation, but feel free to ask questions about them on Tuesday if you have any.

7. In class we showed that there is an infinitely differentiable function

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

   which is nonzero exactly on $(0, \infty)$. Show there exist infinitely differentiable functions with the following properties:* [Discussion continues on the next page.]

   (a) $f_a(x)$ which is nonzero exactly on (a, ∞).

   (b) $g_b(x)$ which is nonzero exactly on $(-\infty, b)$.

   (c) $h_{a,b}(x)$ which is nonzero exactly on (a, b).

   (d) $j_{a,b}(x)$ such that $j_{a,b}(x) = 0$ for $x \geq a$ and $j_{a,b}(x) = 1$ for $x \geq b$.

Note that it follows immediately from the Algebraic Differentiability Theorem that the sum, difference, and (where it makes sense) quotient of infinitely differentiable functions is infinitely differentiable.