Homework 11 Solutions

November 8, 2023

Section 4.4

Problem 4.4.2

(a) No, $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$, either by the $\epsilon - \delta$ argument given in class, or by observing that there is a Cauchy sequence $\left(\frac{1}{n}\right)$ in $(0, 1)$ which is mapped to a sequence $\left(f\left(\frac{1}{n}\right)\right) = (n)$ which is not Cauchy.

(b) Yes, $g(x) = \sqrt{1 + x^2}$ is uniformly continuous on $(0, 1)$. Note that it can be continuously extended to a function $\widetilde{g}$ on $[0, 1]$, for example by taking $\widetilde{g}(x) = \sqrt{1 + x^2}$ on $[0, 1]$.

(c) Yes, $h(x) = x \sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0, 1)$. Note that it can be continuously extended to a function $\widetilde{h}$ on $[0, 1]$, by taking $\widetilde{h}(1) = \sin(1)$ and $\widetilde{h}(0) = 0$.

Problem 4.4.5

Let g be defined on (a, c) such that g is uniformly continuous on $(a, b]$ and on $[b, c)$. Let $\epsilon > 0$, and pick δ_1 such that if $x, y \in (a, b]$ and $|x - y| < \delta_1$, then $|g(x) - g(y)| < \frac{\epsilon}{2}$, and likewise δ_2 such that if $x, y \in [b, c)$ and $|x - y| < \delta_2$ then $|g(x) - g(y)| < \frac{\epsilon}{2}$. Then let $x \leq y$ be any two elements of (a, c), and assume that $|x - y| < \delta = \min\{\delta_1, \delta_2\}$. If either $x \leq y \leq b$ or $b \leq x \leq y$, it follows immediately that $|f(x) - f(y)| < \frac{\epsilon}{2} < \epsilon$. The interesting case is when $x < b < y$. In that case we see that $b - x < y - x < \delta \leq \delta_1$, so since $x, b \in (a, b]$, we have $|f(x) - f(b)| < \frac{\epsilon}{2}$. Likewise since $y - b < y - x < \delta < \delta_2$, so since $y, b \in [b, c)$, we have that $|f(b) - f(y)| < \frac{\epsilon}{2}$. Ergo $|x - y| < \delta$ and $x, y \in (a, c)$ implies that $|f(x) - f(y)| \leq |f(x) - f(b)| + |f(b) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$.

Problem 4.4.6

(a) Possible. Let $f : (0, 1) \to \mathbb{R}$ be $f(x) = \frac{1}{x}$. Then let $(x_n) = \left(\frac{1}{n}\right)$. This is a Cauchy sequence, but $(f(x_n)) = (n)$ is not.

(b) Impossible; as proved in class, the image of a Cauchy sequence in a domain A under a function $f : A \to \mathbb{R}$ which is uniformly continuous is always a Cauchy sequence.

(c) Impossible. Suppose $f : [0, \infty) \to \mathbb{R}$. Let (x_n) be a Cauchy sequence in $[0, \infty)$. Then (x_n) is bounded, hence contained in some $[0, M]$. But f is uniformly continuous on $[0, M]$ because $[0, M]$ is compact, so $(f(x_n))$ is a Cauchy sequence.

Section 4.5

Problem 4.5.2

(a) Possible; let $f : (0, 1) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{4} & 0 < x < \frac{1}{4} \\ x & \frac{1}{4} \leq x \leq \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} < x < 1 \end{cases}$$

so that $f((0, 1)) = \left[\frac{1}{4}, \frac{3}{4}\right]$.

(b) Impossible; a closed interval is compact, and the image of a compact set under a continuous function is compact, hence in particular closed.

(c) Possible; consider $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = (x - 1)^2$, so that $f((0, \infty)) = [0, \infty)$.

(d) Impossible; $\mathbb{R}$ is connected and $\mathbb{Q}$ is not, but the image of a connected set under a continuous function is always connected.

Section 5.2

We set

$$f_a(x) = \begin{cases} x^a & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(a) We claim that f_a is continuous at 0 if $a > 0$. To see this note that $f_a(0) = 0$, so f_a is continuous exactly when $\lim_{x\to 0} f_a(x) = 0$. Now recall that$\lim_{x\to 0} f_a(x) = 0$ is equivalent to the left-hand and right-hand limits of the function at $f_a(x)$ existing and both equalling 0. Since it is always the case that the left-hand limit $\lim_{x\to 0^-} f_a(x) = \lim_{x\to 0^-} 0 = 0$, it suffices to check that $\lim_{x\to 0^+} f_a(x)$ is zero. To the right of zero, $f_a(x) = x^a$, and

$$\lim_{x\to 0^+} x^a = \begin{cases} 0 & a > 0 \\ 1 & a = 0 \\ \infty & a < 0 \end{cases}$$

We conclude that f_a is continuous at 0 if $a > 0$.

(b) We claim that f_a is differentiable at 0 if $a > 1$. For the derivative exists if

$$\lim_{x\to 0} \frac{f_a(x) - f_a(0)}{x - 0} = \lim_{x\to 0} \frac{f_a(x)}{x}$$

exists. Again, this limit exists if the left-hand and right-hand limits are equal, and $\lim_{x\to 0^-} \frac{f_a(x)}{x} = \lim_{x\to 0^-} \frac{0}{x} = 0$. Moreover we see that $\lim_{x\to 0^+} \frac{f_a(x)}{x} = \lim_{x\to 0^+} x^{a-1}$ which by the same logic as in part (a) is equal to 0 exactly when $a-1>0$, or when $a>1$. Then the full derivative function is

$$f_a'(x) = \begin{cases} ax^{a-1} & x>0 \\ 0 & x\le 0 \end{cases}$$

which is continuous at 0 since $a-1>0$.

(c) A close variation on the argument above shows that f_a is twice-differentiable at 0 when $a>2$, and so on.

Other Problems

Problem 4

(a) Let $f(x)=\cos x - x$. We observe that f is continuous on $\left[0,\frac{\pi}{2}\right]$ and $f(0)=1$ whereas $f\left(\frac{\pi}{2}\right)=-\frac{\pi}{2}$. We conclude by the Intermediate Value Theorem that there is some $x\in\left(0,\frac{\pi}{2}\right)$ with the property that $f(x)=0$, or equivalently $x=\cos x$.

(b) Let $g(x)=xe^x-2$. We observe that f is continuous on $[0,1]$ and $g(0)=-2$ whereas $g(1)=e-2>0$. We conclude by the Intermediate Value Theorem that there is some $x\in(0,1)$ with the property that $g(x)=0$, or $xe^x=2$.

Problem 5

Let $p(x)=a_nx^n+\cdots+a_1x+a_0$ be a polynomial of odd degree. Note that $p(x)$ is continuous on $\mathbb{R}$. Without loss of generality let $a_n>0$, since we could multiply $p(x)$ by -1 without changing its roots. Choose $y=\max\{|a_{n-1}|,\ldots,|a_0|\}$ and $x>\max\{1,\frac{ny}{a_n}\}$ so that

$$\begin{aligned} p(x) &= a_nx^n+\cdots+a_1x+a_0 \\ &> a_nx^n-|a_{n-1}|x^{n-1}-\cdots-|a_1|x-|a_0| \\ &> ny(x^{n-1})-|a_{n-1}|x^{n-1}-\cdots-|a_1|x^{n-1}-|a_0|x^{n-1} \\ &> nyx^{n-1}-nyx^{n-1} \\ &= 0 \end{aligned}$$

Similarly choose $z<0$ so that $p(z)<0$. Then there is some $r\in(z,x)$ with the property that $p(r)=0$ by the Intermediate Value Theorem.

Problem 6

(a) We want to compute the derivative of $f(x) = \frac{3x+4}{2x-1}$ at $x = 1$. We see that

$$\begin{aligned} f'(1) &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \to 1} \frac{\frac{3x+4}{2x-1} - 7}{x - 1} \\ &= \lim_{x \to 1} \frac{-11x + 11}{x - 1} \\ &= \lim_{x \to 1} \frac{-11(x - 1)}{x - 1} \\ &= \lim_{x \to 1} -11 \\ &= -11 \end{aligned}$$

(b) We want to compute the derivative of $g(x) = x^2 \cos x$ at $x = 0$. We see that

$$\begin{aligned} g'(0) &= \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{x^2 \cos x - 0}{x - 0} \\ &= \lim_{x \to 0} \frac{x^2 \cos x}{x} \\ &= \lim_{x \to 0} x \cos x \\ &= 0 \end{aligned}$$

(c) We want to compute the derivative of $h(x) = \frac{1}{x}$ at any $c \neq 0$. We see that

$$\begin{aligned} h'(c) &= \lim_{x \to c} \frac{h(x) - h(c)}{x - c} \\ &= \lim_{x \to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} \\ &= \lim_{x \to c} \frac{\frac{c-x}{cx}}{x - c} \\ &= \lim_{x \to c} \frac{-1}{cx} \\ &= -\frac{1}{c^2} \end{aligned}$$