Homework 10 Solutions

November 5, 2023

Section 4.2

Problem 4.2.5

(c) We wish to show that $\lim_{x\to 2}(x^2+x-1)=5$. Let $\epsilon>0$, and set $\delta=\min\{1,\frac{\epsilon}{6}\}$. Assume $0<|x-2|<\delta$. We begin by noting that since $\delta<1$, we have $1<x<3$. We then compute:

$$\begin{aligned}|(x^2+x-1)-5| &= |x^2+x-6|\\ &= |x+3||x-2|\\ &< 6|x-2|\\ &< 6\cdot\frac{\epsilon}{6}\\ &= \epsilon.\end{aligned}$$

As ϵ was arbitrary we are done.

(d) We wish to show that $\lim_{x\to 0}\frac{1}{x}=\frac{1}{3}$. Let $\epsilon>0$, and let $\delta=\min\{1,\frac{6\epsilon}{\}}$.Assume $0<|x-3|<\delta$. We begin by noting that since $\delta<1$, $2<x<4$. We then compute:

$$\begin{aligned}\left|\frac{1}{x}-\frac{1}{3}\right| &= \left|\frac{x-3}{3x}\right|\\ &= \frac{|x-3|}{3x}\\ &< \frac{|x-3|}{6}\\ &< \frac{6\epsilon}{6}\\ &= \epsilon.\end{aligned}$$

As ϵ was arbitrary we are done.

Problem 4.2.8

(a) Let $f(x)=\frac{|x-2|}{x-2}$. Consider the sequence $x_n=2-\frac{1}{n}$, so that $x_n\to 2$. We see that $f(x_n)=\frac{|x_n-2|}{x_n-2}=-1$ for all n, so $f(x_n)\to -1$. But if we instead consider $y_n=2+\frac{1}{n}$, we have that $y_n\to 2$ but $f(y_n)=1$ for all n so $f(y_n)\to 1$. Therefore the limit $\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

(b) We now consider $\lim_{x\to\frac{7}{4}} \frac{|x-2|}{x-2}$. Let $\epsilon > 0$. Let $\delta = \frac{1}{4}$. Then, in particular, if $0 < |x - \frac{7}{4}| < \delta$, we have that $x < 2$, so that $f(x) = -1$. In particular, if $0 < |x - \frac{7}{4}| < \delta$ then $|f(x) - (-1)| = 0 < \epsilon$. Ergo, $\lim_{x\to\frac{7}{4}} \frac{|x-2|}{x-2} = -1$.

(c) Let $f(x) = (-1)^{[[\frac{1}{x}]]}$. Consider the sequence $\left(\frac{1}{n}\right)$ converging to 0. We see that $f\left(\frac{1}{n}\right) = (-1)^n$. This does not converge. So $\lim_{x\to 0} f(x)$ does not exist.

(d) Let $f(x) = x^{\frac{1}{3}} \cdot (-1)^{[[\frac{1}{x}]]}$. Then given $\epsilon > 0$, let $\delta = \epsilon^3$. Then if $0 < |x| < \delta$, we have that $|f(x) - 0| = |f(x)| = |x|^{\frac{1}{3}} < \epsilon$. Hence $\lim_{x\to 0} f(x) = 0$.

Problem 4.2.10

(a) Let $f : A \to \mathbb{R}$ be a function, and let c be a limit point of $A \cap \{x : x < a\}$. Then we say that $\lim_{x\to a^-} f(x) = M$ if for any $\epsilon > 0$, it is the case that $a - \delta < x < a$ and $x \in A$ implies that $|f(x) - M| < \epsilon$. Similarly if a is a limit point of $A \cap \{x : x > a\}$, we say that $\lim_{x\to a^+} f(x) = L$ if for any $\epsilon > 0$, it is the case that $a < x < a + \delta$ and $x \in A$ implies that $|f(x) - L| < \epsilon$.

(b) Assume that $f : A \to \mathbb{R}$ is a function, and a is a limit point of both $A \cap \{x : x < a\}$ and $A \cap \{x : x > a\}$. (If it isn't, this question doesn't actually quite make sense.)
First assume $\lim_{x\to a} f(x) = L$. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ and $x \in A$ implies that $|f(x) - L| < \epsilon$. In particular this means that $a - \delta < x < a$ and $x \in A$ implies $|f(x) - L| < \epsilon$, so $\lim_{x\to a^-} f(x) = L$. Similarly it also means that $a < x < a + \delta$ and $x \in A$ implies $|f(x) - L| < \epsilon$, so $\lim_{x\to a^+} f(x) = L$
In the other direction, assume that $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$. Let $\epsilon > 0$. Then there is a $\delta_1 > 0$ such that $a - \delta_1 < x < a$ and $x \in A$ implies that $|f(x) - L| < \epsilon$. Likewise there is a δ_2 such that $a < x < a + \delta_2$ and $x \in A$ implies that $|f(x) - L| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2$, then $0 < |x - a| < \delta$ and $x \in A$ implies that $|f(x) - L| < \epsilon$. So $\lim_{x\to a} f(x) = L$.
Remark: The second implication depends, very heavily, on there being a finite number of directions (two) from which we can approach a. It's not true for limits on the plane $\mathbb{R}^2$, for example.

Section 4.3

Problem 4.3.1

(a) Let $g(x) = x^{\frac{1}{3}}$. Given $\epsilon > 0$, let $\delta = \epsilon^3$. Then if $|x| = |x - 0| < \delta$, we have that $|x^{\frac{1}{3}} - 0| = |x|^{\frac{1}{3}} < \epsilon$. Ergo g is continuous at 0.

(b) Let $c \neq 0$. Given $\epsilon > 0$, let $\delta < \min\{c^{\frac{2}{3}}\epsilon, |c|\}$, so that in particular if $|x - c| < \delta$ x and c have the same sign. Then if $|x - c| < \delta$, we have that

$$\begin{aligned}
|x^{\frac{1}{3}} - c^{\frac{1}{3}}| &= |x^{\frac{1}{3}} - c^{\frac{1}{3}}| \cdot \frac{|x^{\frac{2}{3}} + c^{\frac{1}{3}}x^{\frac{1}{3}} + c^{\frac{2}{3}}|}{|x^{\frac{2}{3}} + c^{\frac{1}{3}}x^{\frac{1}{3}} + c^{\frac{2}{3}}|} \\
&= \frac{|x - c|}{|x^{\frac{2}{3}} + c^{\frac{1}{3}}x^{\frac{1}{3}} + c^{\frac{2}{3}}|} \\
&= \frac{|x - c|}{x^{\frac{2}{3}} + c^{\frac{1}{3}}x^{\frac{1}{3}} + c^{\frac{2}{3}}} \\
&< \frac{|x - c|}{c^{\frac{2}{3}}} \\
&< \frac{c^{\frac{2}{3}}\epsilon}{c^{\frac{2}{3}}} \\
&= \epsilon.
\end{aligned}$$

Problem 4.3.4

(a) Let $f(x) \equiv 1$, and let

$$g(x) = \begin{cases} 2 & x \neq 1 \\ 0 & x = 1 \end{cases}$$

such that $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 1} g(x) = 2$, but $\lim_{x\to 0} g(f(x)) = 0$.

(b) If we assume that f and g are continuous on $\mathbb{R}$, then we have $\lim_{x\to p} f(x) = f(p)$ and $\lim_{x\to f(p)} g(x) = g(f(p))$, and from the fact that the composition of continuous functions is continuous we see that $\lim_{x\to p} g(f(x)) = g(f(p))$, so the relationship between the limits is true.

(c) We can get the result of (a) even if the function f is continuous; consider the example above. But not suppose that g is continuous (in particular, continuous at q) and we have $\lim_{x\to p} f(x) = q$ and $\lim_{x\to q} g(x) = r = g(q)$. Then if x_n is a sequence of points with $x_n \neq p$ and $x_n \to p$, we have that $f(x_n) \to q$, so since g is continuous at q, we see that $g(f(x_n)) \to g(q)$. As (x_n) was arbitrary we observe that $\lim_{x\to p} g(f(x)) = g(q) = r$.

Problem 4.3.6

For the statements below, we assume that f and g have the same domain and 0 is a limit point of the domain.

(a) Let

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

and

$$g(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$$

so that neither f nor g is continuous at 0 but $fg(x) \equiv 0$ and $f + g(x) \equiv 1$ both are.

(b) Impossible. Suppose we have the situation that $f(x)$ and $f(x)+g(x)$ are continuous at 0. Then let (x_n) be any sequence of points converging to 0 in the mutual domain of the three functions. We have that $g(x_n) = (f(x_n)+g(x_n)) - f(x_n) \to (f(0)+g(0)) - f(0) = g(0)$ using continuity of f and $f+g$ at 0 and the Algebraic Limit Theorem. But since (x_n) was arbitrary, g is in fact continuous at 0.

(c) Let $f(x) \equiv 0$ be the zero function, and $g(x)$ be any function not continuous at 0.

(d) Let

$$f(x) = \begin{cases} 2 & x \leq 0 \\ \frac{1}{2} & x > 0 \end{cases}$$

so that $g(x) = f(x) + \frac{1}{f(x)} \equiv \frac{3}{2}$ for all x.

(e) Impossible. If $h(x) = [f(x)]^3$ is continuous at 0, recall from Problem 4.3.1 that $g(x) = x^{\frac{1}{3}}$ is continuous on $\mathbb{R}$, and therefore in particular at $[f(0)]^3$. The composition of continuous functions is continuous, so $f(x) = g(h(x))$ is continuous at 0.

Section 4.4

Problem 4.4.8

(a) Impossible; the image of a compact set under a continuous function is compact.

(b) Possible; let

$$f(x) = \begin{cases} 0 & 0 < x < \frac{1}{4} \\ 4x-1 & \frac{1}{4} < x < \frac{1}{2} \\ 1 & \frac{1}{2} < x \end{cases}$$

(c) Let

$$g(x) = \frac{\left|\sin\left(\frac{1}{x}\right)\right| + x}{1+2x}$$

on $[0,1)$. We observe that because all four terms in the expression are positive, this is always a positive number; moreover, since $\left|\sin\left(\frac{1}{x}\right)\right| + x \leq 1 + x < 1 + 2x$, we see that $g(x) < 1$. Since the image of an interval is an interval, to show that $g((0,1])$ is $(0,1)$ it suffices to check that we can find an x such that $g(x) < \epsilon$ for all $\epsilon > 0$ and a y such that $g(y) > 1-\epsilon$ or equivalently $1 - g(y) < \epsilon$ likewise for all $\epsilon > 0$.

So, let $\epsilon > 0$. Then pick $x = \frac{1}{2\pi n} < \epsilon$. We have $g(x) = \frac{0+x}{1+2x} < x < \epsilon$. Similarly if we pick y such that $y = \frac{1}{2\pi n + \frac{\pi}{2}} < \epsilon$, we have that $1 - g(y) = 1 - \frac{1+y}{1+2y} = \frac{y}{1+2y} < y < \epsilon$. So $g((0,1]) = (0,1)$.

Problem 4.4.12

(a) False. Let $f : \mathbb{R} \to \mathbb{R}$ be the constant function $f(x) = 0$ for all $x \in \mathbb{R}$. Certainly f is continuous on $\mathbb{R}$. Then $\{0\}$ is finite but $f^{-1}(\{0\}) = \mathbb{R}$ is not.

(b) False, by the same example as (a); $\{0\}$ is compact but $\mathbb{R}$ is not.

(c) False, again by the same example.