Math 311H
Honors Introduction to Real Analysis

Final

Instructions: You have three hours to complete the exam. There are nine questions, worth a total of forty-five points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ________________________________

Question	Points	Score
1	4	
2	5	
3	5	
4	4	
5	5	
6	5	
7	5	
8	6	
9	6	
Total:	45	

1. For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.

   (a) [1pts.] A subset $E \subset \mathbb{R}$ whose limit points are exactly the rational numbers $\mathbb{Q}$.

   Solution: Impossible. Let L be the set of limit points of E. Recall that any limit point of L is also a limit point of E. But the set of limit points of $\mathbb{Q}$ is all of $\mathbb{R}$, so any set whose limit points include all the rationals in fact has a limit point at every real number.

   (b) [1pts.] A power series that converges uniformly on its interval of convergence.

   Solution: An example is $\sum_{n=1} \frac{x^n}{n^2}$, which converges uniformly on $[-1, 1]$.

   (c) [1pts.] A point $x \in \left(0, \frac{\pi}{2}\right)$ such that $x \geq \tan x$.

   Solution: This is not possible. Let $f(x) = \tan x - x$. Observe that $f(0) = 0$ and $f'(x) = \sec^2(x) - 1$, which is positive on $\left[0, \frac{\pi}{2}\right)$. Hence f is increasing on this interval, so that $f(x) \geq 0$ on $\left(0, \frac{\pi}{2}\right)$, or equivalently $\tan x \geq x$.

   (d) [1pts.] A nonempty connected set which contains no nonempty compact subset.

   Solution: This is not possible; every nonempty set contains a compact subset because one-element sets are compact.

2. [5pts.] Let (f_n) be a sequence of functions on an interval (a, b) such that each f_n is uniformly continuous on (a, b). Suppose that $f_n \to f$ uniformly on (a, b). Prove or disprove: f is also uniformly continuous on (a, b).

   Solution: The statement is true. Let $\epsilon > 0$. Pick N such that $n \geq N$ implies that $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in (a, b)$. Furthermore, choose δ such that $|x - y| < \delta$ implies that $|f_N(x) - f_N(y)| < \frac{\epsilon}{3}$. Then we see that for $|x - y| < \delta$ and $x, y \in (a, b)$, we have

   $$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

   Ergo f is uniformly continuous on (a, b).

3. (a) [3pts.] Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with the property that infinitely many a_n are integers. Prove that the series must have radius of convergence $R \leq 1$.

Solution: Suppose $R > 1$. Then $\sum_{n=0}^{\infty} a_n(1)^n$ converges absolutely. In particular $\sum_{n=0}^{\infty} |a_n|$ converges, so $|a_n| \to 0$, so there is some N such that $n \geq N$ implies that $|a_n| < 1$. This shows that only finitely many a_n can be integers. We conclude that if infinitely many a_n are integers, $R \leq 1$.

(b) [2pts.] Give an example of a power series of the form above with $R = 1$.

Solution: The geometric series $\sum_{n=0}^{n} x^n$ is a fine example.

4. [4pts.] A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a fixed point if $f(x) = x$. Prove that if f is differentiable on A with $f'(x) \neq 1$ for all x, then f has at most one fixed point on A.

Solution: Since f is differentiable on A, we have that f is also continuous on A. Suppose that f has two fixed points $x < y$ in A, and apply the Mean Value Theorem to $[x, y]$. Then there is some $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x} = \frac{y - x}{y - x} = 1$$

which contradicts the assumption that $f' \neq 1$ on A. So f has at most one fixed point.

5. (a) [3pts.] Compute $\sum_{n=2}^{\infty} \frac{n^2}{3^n}$.

Solution: We recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ on $(-1, 1)$. Power series are differentiable term-by-term so we have

$$\frac{1}{(1-x)^2} = \sum_{n=1} nx^{n-1} = 1 + 2x + 3x^2 + \ldots$$

and therefore

$$\frac{x}{(1-x)^2} = \sum_{n=1} nx^n = x + 2x^2 + 3x^3 + \ldots$$

all on $(-1, 1)$. Differentiating again, we see that on $(-1, 1)$, we have

$$\frac{1(1-x)^2 - x(-2)(1-x)}{(1-x)^4} = \frac{1 - x^2}{(1-x)^4} = 1 + 4x + 9x^2 + \ldots.$$

We multiply by x to see that on $(-1, 1)$, we have

$$\frac{x(1-x^2)}{(1-x)^4} = x + 4x^2 + 9x^3 + \cdots = \sum_{n=1}^{\infty} n^2 x^n.$$

We see that

$$\sum_{n=2}^{\infty} \frac{n^2}{3^n} = \frac{(1/3)(1-(1/3)^2)}{(1-(1/3))^4} - \frac{1}{3}$$
$$= \frac{3}{2} - \frac{1}{3}$$
$$= \frac{7}{6}$$

(b) [2pts.] Estimate $\frac{1}{e}$ to within $\frac{1}{100}$.

Solution: Using the Taylor series expansion of e^x we have that

$$\frac{1}{e} = e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \dots$$

As this is alternating the error on any given partial sum is bounded by the absolute value of the next term of the sequence. We have $\frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$, so our estimation is

$$e^{-1} \sim 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{3}{8}.$$

6. Consider the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$.

(a) [2pts.] At what points does the series converge? Is the convergence conditional or absolute?

Solution: We observe that the absolute values of the terms of the sequence are $|a_n| = \frac{x^2+n}{n^2} = \frac{x^2}{n^2} + \frac{1}{n}$. These are decreasing to zero for any given $x \in \mathbb{R}$, and the series is alternating, so it converges everywhere. However, $|a_n| > \frac{1}{n}$, so the series does not converge absolutely at any point, only conditionally.

(b) [3pts.] Prove that the series converges uniformly on every bounded interval. [Hint: Consider derivatives.]

Solution: We differentiate term-by-term to obtain the series $\sum_{n=1}^{\infty}(-1)^n \frac{2x}{n^2}$. Let us restrict to a bounded interval (a, b). Since

$$\left|(-1)^n \frac{2x}{n^2}\right| = (2x)\frac{1}{n^2} < 2\max\{|a|, |b|\}\frac{1}{n^2} = M_n$$

and $\sum_{n=1}^{\infty} M_n$ converges, by the Weierstrass M-test we see that $\sum_{n=1}^{\infty}(-1)^n \frac{2x}{n^2}$ converges uniformly on every bounded interval. By the Differentiable Limit

Theorem, since the original series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ converges at at least one point in every bounded interval, it also converges uniformly.

Note that attempting to use the Weierstrass M-Test on the original series does not work, as it only detects uniform convergence of series which are pointwise absolutely convergent.

7. Compute the derivative functions of the following functions, where they exist.

(a) [3pts.] $g(x) = xe^{|x|}$

Solution:
We may rewrite this function as

$$g(x) = \begin{cases} xe^{-x} & x < 0 \\ xe^{x} & x > 0 \end{cases}$$

The interesting case is when $x = 0$. We see that the lefthand limit is $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{xe^{-x}}{x} = \lim_{x\to 0^-} e^{-x} = 1$ whereas the righthand limit is $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{xe^{x}}{x} = \lim_{x\to 0^+} = 1$. We conclude that

$$g'(x) = \begin{cases} e^{-x} - xe^{-x} & x < 0 \\ 1 & x = 0 \\ e^{x} + xe^{x} & x > 0 \end{cases}$$

which we may repackage as $g'(x) = (1 + |x|)e^{|x|}$.

(b) [2pts.]

$$f(x) = \begin{cases} x \sin x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Solution: Let c be any real number. Recall that there is a sequence of rational numbers (x_n) such that $x_n \to c$ and a sequence of irrational numbers (y_n) such that $y_n \to c$. We observe that $f(x_n) = x_n \sin(x_n) \to c \sin c$ and $f(y_n) = 0 \to 0$, so f is discontinuous for any c such that $c \sin(c) \neq 0$. Since differentiability implies continuity we need only investigate points where $c \sin c = 0$, or where $c = n\pi$ for some integer n.

For $c = n\pi$, the derivative at c if it is exists is $\lim_{x\to c} \frac{f(x)-c\sin(c)}{x-c}$. If $n \neq 0$, if we approach along the sequence (x_n) this quotient is

$\frac{x_n \sin(x_n) - c\sin(c)}{x-c} = \sin(c) + c\cos(c) = c$ and if we approach along the sequence (y_n) this quotient is $\frac{0-0}{x-c} = 0 \to 0$. So, f is not differentiable at $c = n\pi$ for $n \neq 0$.

However, at $c = 0$, the derivative is $\lim_{x\to 0} \frac{f(x)}{x}$ if it exists, and $0 \le \left|\frac{f(x)}{x}\right| \le |\sin x|$, so by the Squeeze Theorem for Functional Limits we conclude that $f'(0) = 0$.

8. Compute the following limits.
   (a) [2pts.] $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Solution: We multiply both the numerator and denominator by $\sqrt{1+x} + \sqrt{1-x}$, obtaining

$$\begin{aligned}
\lim_{x\to 0} \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} &= \lim_{x\to 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})} \\
&= \lim_{x\to 0} \frac{2}{(\sqrt{1+x}+\sqrt{1-x})} \\
&= \frac{2}{2} \\
&= 1
\end{aligned}$$

   (b) [2pts.] $\lim_{x\to 0^+} x^x$

Solution: We replace the limit with

$$\begin{aligned}
\lim_{x\to 0^+} x^x &= \lim_{x\to 0^+} e^{\ln(x^x)} \\
&= \lim_{x\to 0^+} e^{x\ln(x)}
\end{aligned}$$

We observe that the term in the exponent has

$$\begin{aligned}
\lim_{x\to 0^+} x\ln(x) &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to 0^+} -x \\
&= 0
\end{aligned}$$

where the second equality is by L'Hospital's Rule since the limit on the right exists. So since e^x is a continuous function, the original limit is $e^0 = 1$.

(c) [2pts.] $\lim_{x\to 0} \frac{1-\cos x}{e^x - 1}$

Solution: We observe that the numerator and denominator evaluate to 0 at 0, so the limit is equal to
$$\lim_{x\to 0} \frac{\sin x}{e^x} = \frac{0}{1} = 0$$
by L'Hospital's Rule since the latter exists.

9. A sequence of functions $f_n \colon A \to \mathbb{R}$ is said to be *equicontinuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x - y| < \delta$ implies that $|f_n(x) - f_n(y)| < \epsilon$ for all $x, y \in A$ and all n.

(a) [2pts.] Give an example of a pointwise convergent sequence of functions $f_n \colon A \to \mathbb{R}$ such that each f_n is uniformly continuous on A but (f_n) is not equicontinuous on A.

Solution: Consider the sequence
$$f_n(x) = \begin{cases} n^2 x & 0 \le x \le \frac{1}{n} \\ 2n - n^2 x & \frac{1}{n} < x \le \frac{2}{n} \\ 0 & \frac{2}{n} < x \le 1 \end{cases}$$
Each of these functions is continuous on $[0,1]$, hence uniformly continuous. But given $\epsilon = 1$ and any $\delta > 0$ we may choose N such that $\frac{1}{N} < \delta$. Then $\left|0 - \frac{1}{N}\right| < \delta$ but $|f_N(0) - f_N\left(\frac{1}{N}\right)| = |0 - N| = N > 1$. So (f_n) is not equicontinuous.

(b) [2pts.] Let (f_n) with $f_n \colon [0,1] \to \mathbb{R}$ be equicontinuous and uniformly bounded; that is, there exists M with the property that $|f_n(x)| \le M$ for all $x \in [0,1]$ and all n. Prove (f_n) has a subsequence which converges pointwise at every rational number. [Hint: By Bolzano-Weierstrass, there is certainly a subsequence of $(f_n(1))$ which converges. How could you modify this to converge at a second rational?]

Solution: Consider any enumeration of the rational numbers $\{r_1, r_2, \dots\}$. The sequence $(f_n(r_1))$ is bounded, hence has a convergent subsequence $(f_{n_{1,k}}(r_1))$. Now, if we consider the sequence of functions $(f_{n_{1,k}})$ and apply them to r_2 we see there is a subsequence of $(f_{n_{1,k}}(r_2))$ which converges, call it $(f_{n_{2,k}}(r_2))$. Indeed, we may insist that $f_{n_{2,1}} = f_{n_{1,1}}$, that is, we did not discard the first term in the sequence. Then $(f_{n_{2,1}})$ converges at both r_1 and r_2. Continue cutting down the sequence in this way, such that at the jth step we keep the first $j-1$ terms of the subsequence $f_{n_{1,1}}, f_{n_{2,2}}, \dots, f_{n_{j-1,j-1}}$ and cut the rest down to a subsequence also converging at r_j. The resulting subsequence $(f_{n_{j,j}})$ converges at every rational as desired.

(c) [2pts.] Prove that the subsequence of part (b) converges uniformly on all of $[0, 1]$. [Hint: $[0, 1]$ may be covered by finitely many neighborhoods of length δ for any δ.]

Solution: Rename the subsequence from part (b) to be (f_n), since we no longer care about the original sequence. Let $\epsilon > 0$. Choose δ such that for $x, y \in [0, 1]$, we have that $|x-y| < \delta$ implies that $|f_n(x)-f_n(y)| < \frac{\epsilon}{3}$ for all n. Now cover $[0, 1]$ by finitely many neighborhoods $V_\delta(r_1), V_\delta(r_2), \ldots, V_\delta(r_k)$ where $\{r_1, \ldots, r_k\}$ are rationals. Now, for each r_i, $(f_n(r_i))$ converges as a sequence of real numbers and therefore satisfies the Cauchy condition, so there exists N_i such that for $n, m \geq N_i$, we have $|f_n(r_i) - f_n(r_i)| < \frac{\epsilon}{3}$. Let $N = \max\{N_1, \ldots, N_k\}$.

Now let $x \in [0, 1]$. Then x lies in some $V_\delta(r_i)$, and $|x - r_i| < \delta$. So for $n, m \geq N$, we have

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(r_i)| + |f_n(r_i) - f_m(r_i)| + |f_m(r_i) - f_m(x)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

As x was arbitary, we see that (f_n) satisfies the Cauchy condition on $[0, 1]$ and therefore converges uniformly.

This result is called the Arzela-Ascoli Theorem.