Math 311H
Honors Introduction to Real Analysis

Sample Final

Instructions: You have three hours to complete the exam. There are nine questions, worth a total of forty-five points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ________________________________

Question	Points	Score
1	4	
2	4	
3	5	
4	5	
5	6	
6	5	
7	6	
8	5	
9	5	
Total:	45	

1. For each of the following things, either give an example of the described object (no need to justify it) or write a brief explanation of why this is impossible.

   (a) [1pts.] A power series with interval of convergence $(c-R, c+R]$ which converges absolutely on the entire interval.

   Solution: This is impossible. If a power series converges absolutely at any $c+\ell$, then it converges absolutely on $[c-|\ell|, c+|\ell|]$.

   (b) [1pts.] A compact set which contains no nontrivial interval.

   Solution: The Cantor set is an example.

   (c) [1pts.] A function $f(x)$ which is differentiable on all of $\mathbb{R}$ with $f'(x) < 4$ for all x and two points $a, b \in [2, \infty)$ with the property that $f(a) = a^2$ and $f(b) = b^2$.

   Solution: This is impossible. Since f is differentiable on $\mathbb{R}$, it follows that f is also continuous on $\mathbb{R}$. Suppose there are two values $a < b$ such that $f(a) = a^2$ and $f(b) = b^2$. By the Mean Value Theorem, there would then be a point $c \in (a, b)$ such that
   $$\frac{f(b)-f(a)}{b-a} = f'(c).$$
   Substituting in we have
   $$\frac{b^2-a^2}{b-a} = f'(c) \leq 4$$
   so we see that in this case $b + a \leq 4$. Hence it is not the case that both a and b are greater than 2.

   (d) [1pts.] An infinite subset S of $[0, 1]$ with no limit point in $[0, 1]$.

   Solution: This is impossible. If S is infinite, we may construct a sequence of elements (s_k) of S such that no element is repeated. As this sequence is bounded, it has a convergent subsequence (s_{n_k}) with limit some s. After deleting at most one appearance of s from (s_{n_k}), this is a sequence of points in S none of which is s converging to s, so we see thath s is a limit point of S. As $[0, 1]$ is closed and (s_{n_k}) is also a sequence in $[0, 1]$, we see that $s \in [0, 1]$.

2. [4pts.] Describe all of the functions f which are solutions to the differential equation $f'' = -f$ and may be represented by a power series on some interval about $c = 0$.

   Solution: Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on some interval about 0. Then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Since Taylor's formula for the coefficients implies that the power series representing a function on an interval is unique, if $-f = f''$ we must have $-a_{n-2} = n(n-1)a_n$ for all $n \geq 2$, or rephrased $a_n = -\frac{a_{n-2}}{n(n-1)}$. For example, $a_2 = -\frac{a_0}{2(1)}$, and $a_4 = -\frac{a_2}{4(3)} = \frac{a_0}{4!}$. Inductively we see that the coefficients depend only on the value of n modulo 4, with

$$\begin{aligned}
f(x) &= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 - \dots \\
&= \sum_{n=0}^{\infty}\left[a_0\frac{(-1)^n x^{2n}}{(2n)!} + a_1\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] \\
&= a_0\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} + a_1\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
&= a_0\cos x + a_1 \sin x
\end{aligned}$$

Therefore, all of the functions representable by a power series about 0 with $-f = f''$ are of the form $f(x) = a_0 \cos x + a_1 \sin x$ for some $a_0, a_1 \in \mathbb{R}$.

3. [5pts.] Suppose that f is a differentiable function on an interval A with the property that $|f'(x)| \leq M$ on A. Prove that f is uniformly continuous on A.

Solution: Let $\epsilon > 0$, and let $\delta = \frac{\epsilon}{M}$. Suppose that $x, y \in A$ such that $|x - y| < \delta$. Then by the Mean Value Theorem there is some $c \in (x, y)$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(c)$$

Taking the absolute value of both sides we see that $\frac{|f(x)-f(y)|}{|x-y|} = |f'(c)| \leq M$, or $|f(x) - f(y)| \leq M|x - y| < \epsilon$. Since $\epsilon > 0$ was arbitrary we are done.

4. Compute the derivative functions of the following functions where they exist.
   (a) [2pts.] $f(x) = |x| + |x - 1|$

Solution: We see that this function may be rewritten

$$f(x) = \begin{cases} 1 - 2x & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 2x - 1 & 1 < x \end{cases}$$

We conclude that

$$f(x) = \begin{cases} -2 & x < 0 \\ 0 & 0 < x < 1 \\ 2 & 1 < x \end{cases}$$

and fails to exist at 0 and 1.

(b) [3pts.]

$$g(x) = \begin{cases} \left(\sin^2 x\right) \cdot \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Solution: The interesting case is $x = 0$. We see that

$$\begin{aligned} \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^+} \frac{\sin^2(x) \sin\left(\frac{1}{x}\right)}{x} \\ &= \lim_{x \to 0^+} \left(\frac{\sin(x)}{x}\right) \cdot \sin x \cdot \sin\left(\frac{1}{x}\right). \end{aligned}$$

As $x \to 0$, we have that $\frac{\sin x}{x} \to 1$, the term $\sin x \to 0$, and $\sin\left(\frac{1}{x}\right)$ is bounded. So the entire limit is zero. Hence the derivative function is

$$g'(x) = \begin{cases} 2 \sin x \cos x \sin\left(\frac{1}{x}\right) - \frac{\sin^2(x)}{x^2} \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

5. Compute the following limits.

(a) [2pts.] $\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

Solution: We replace the limit with

$$\lim_{x \to 0} e^{\ln((\cos x)^{\frac{1}{x^2}})}$$

which becomes

$$\lim_{x \to 0} e^{\frac{\ln(\cos x)}{x^2}}$$

The limit of the exponent is

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(\cos x)}{x^2} &= \lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{2x}\\ &= \lim_{x\to 0} -\frac{\sin x}{2x\cos x}\\ &= \lim_{x\to 0} -\frac{1}{2\cos x}\cdot\frac{\sin x}{x}\\ &= -\frac{1}{2}(1)\\ &= -\frac{1}{2}\end{aligned}$$

where the second step is an application of L'Hospital's Rule. So the entire limit is $e^{-\frac{1}{2}}$, or $\frac{1}{\sqrt{e}}$.

(b) [2pts.] $\lim_{x\to 0}\frac{\tan x - x}{x^3}$

Solution: We observe that the numerator and denominator both evaluate to zero at zero, so we know by L'Hospital's Rule that the limit above is equal to

$$\lim_{x\to 0}\frac{\sec^2(x)-1}{3x^2}$$

if it exists. But the numerator and denominator are still continuous and evaluate to zero at zero, so this second limit is equal to

$$\lim_{x\to 0}\frac{2\sec^2(x)\tan x}{6x}$$

which after yet another application of the same principle becomes

$$\lim_{x\to 0}\frac{2\sec^2(x)\tan^2 x + 2\sec^4(x)}{6} = \frac{0+2}{6} = \frac{1}{3}.$$

(c) [2pts.] $\lim_{x\to 0}\frac{1}{e^x-1}-\frac{1}{x}$

Solution: We rewrite the limit as

$$\lim_{x\to 0}\frac{x-e^x-1}{xe^x}.$$

We observe that the numerator and denominator are both continuous and evaluate to zero at zero, so we know by L'Hospital's Rule that the limit above is equal to

$$\lim_{x\to 0} \frac{1-e^x}{e^x + xe^x}$$

which by the same principle is equal to

$$\lim_{x\to 0} \frac{-e^x}{e^x + e^x + xe^x} = -\frac{1}{2}.$$

6. (a) [4pts.] Suppose that $K \subset \mathbb{R}$ is compact, and $f_n : K \to \mathbb{R}$ is a sequence of continuous functions such that $f_n \to f$ pointwise, and such that $f_n(x) \le f_{n+1}(x)$ for all $x \in K$ and f is continuous. Show that in fact (f_n) converges uniformly.

   [Hint: For $\epsilon > 0$, let K_n be the set of $x \in K$ for which $f(x) - f_n(x) \ge \epsilon$ and consider the sets $K_1, K_2, \ldots$]

   Solution: First we note that since each $(f_n(x))$ is an increasing sequence, $f_n(x) \le f(x)$ for all n and $x \in K$. For each n, let K_n be the set of $x \in K$ such that $f(x) - f_n(x) \ge \epsilon$. Because of monotonicity, we see that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ is a nested sequence of sets. Moreover, K_n is closed since it is $(f - f_n)^{-1}([\epsilon, \infty))$, and f_n and f are both continuous so their difference is as well, and the preimages of closed sets under continuous functions are closed. As K is bounded, K_n must be bounded, so K_n is compact. Therefore, if K_n is nonempty for all n, we have $\cap_{n=1}^{\infty} K_n \ne \emptyset$. But this is a contradiction, since for any $x \in K$ there is some N such that $n \ge N$ implies that $|f(x) - f_n(x)| < \epsilon$ and therefore $x \notin K_n$ for $n \ge N$. So, there must exist some N' for which $K_{N'}$ is empty, and then for $n \ge N'$, K_n is also empty. This implies that $|f(x) - f_n(x)| < \epsilon$ for all $x \in K$ and $n \ge N'$. So the convergence is uniform.

   This result is called Dini's Theorem.

   (b) [1pts.] Give an example to show that compactness is necessary in the proposition above. Your example can be either increasing or decreasing; the proposition above works for monotone generally.

   Solution: The sequence $f_n(x) = \frac{x}{n}$ is monotone on $[0, \infty)$, each f_n is continuous, and (f_n) converges pointwise to the continuous function $f \equiv 0$, but the convergence is not uniform. (For example, if $\epsilon = 1$, it suffices to note that $f_N(N) = 1$ is not within distance ϵ of f.)

7. Consider the sequence of functions $f_n(x) = \frac{x^n}{1+x^n}$.

   (a) [2pts.] What is the pointwise limit of (f_n) on $[0, \infty)$?

Solution: We see that the pointwise limit is

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ \frac{1}{2} & x = 1 \\ 1 & x \in (1,\infty) \end{cases}$$

(b) [2pts.] Does (f_n) converge uniformly on $[0,1]$?

Solution: No; the uniform limit of continuous functions is continuous, but f is not continuous at $x = 1$.

(c) [2pts.] Does (f_n) converge uniformly on $[2,\infty)$?

Solution: Yes, for $x \geq 2$ we have

$$|f(x) - f_n(x)| = \left|1 - \frac{x^n}{1+x^n}\right| = \frac{1}{1+x^n} \leq \frac{1}{1+2^n} < \frac{1}{2^n}.$$

So, given $\epsilon > 0$, it suffices to choose N such that $\frac{1}{2^N} < \epsilon$ so that $n \geq N$ implies that $|f(x) - f_n(x)| < \epsilon$.

8. (a) [3pts.] Compute $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$.

Solution: We recall that on $(-1,1)$ we have $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. Power series are differentiable term by term with the result having the same radius of convergence, so it follows that

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

and that

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

for $x \in (-1,1)$. In paricular we now have

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} \frac{-1/3}{(4/3)^2} = -\frac{3}{4}.$$

(b) [2pts.] Estimate $\sin(.2)$ to within $\frac{1}{1000}$.

Solution: We recall that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$. This is an alternating sequence, so the error of any given partial sum as an approximation to the total is bounded above by the absolute value of the next term. We observe that $\frac{(.2)^5}{5!} = \frac{.00032}{120}$ is less than $\frac{1}{1000}$, so our approximation is $\sin(.2) \sim .2 - \frac{.2^3}{3!} = \frac{1192}{6000}$.

9. A sequence of functions (f_n) with $f_n \colon A \to \mathbb{R}$ is said to be *compactly convergent* if, for every compact set $K \subset A$, the sequence $f_n \colon K \to \mathbb{R}$ converges uniformly.

   (a) [2pts.] Give an example of a sequence of functions $f_n \colon A \to \mathbb{R}$ with the property that (f_n) is compactly convergent but not uniformly convergent.

   Solution: For example $p_n(x) = 1 + x + \cdots + x^n$ the partial sums of the geometric series are uniformly convergent on compact sets within their domain but are not uniformly convergent on the entire interval $(-1, 1)$ on which they converge pointwise.

   (b) [3pts.] Prove that if (f_n) converges compactly on a domain A and each f_n is continuous at some $c \in A$, then the limit f is continuous at c. Remark: Note that we are not assuming A contains any interval.

   Solution: Let (a_n) be a sequence of points in A converging to c. Then $K = \{a_n : n \in \mathbb{N}\} \cup \{c\}$ is closed and bounded, hence compact. So (f_n) converges uniformly on K, implying that f is continuous on K, so in particular $f(a_n) \to f(c)$. As (a_n) was arbitrary we are done.