Math 311H
Honors Introduction to Real Analysis

Sample Final

Instructions: You have three hours to complete the exam. There are nine questions, worth a total of forty-five points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

Question	Points	Score
1	4	
2	4	
3	5	
4	5	
5	6	
6	5	
7	6	
8	5	
9	5	
Total:	45	

1. For each of the following things, either give an example of the described object (no need to justify it) or write a brief explanation of why this is impossible.

   (a) [1pts.] A power series with interval of convergence $(c - R, c + R]$ which converges absolutely on the entire interval.

   (b) [1pts.] A compact set which contains no nontrivial interval.

   (c) [1pts.] A function $f(x)$ which is differentiable on all of $\mathbb{R}$ with $f'(x) < 4$ for all x and two points $a, b \in [2, \infty)$ with the property that $f(a) = a^2$ and $f(b) = b^2$.

   (d) [1pts.] An infinite subset S of $[0, 1]$ with no limit point in $[0, 1]$.

2. [4pts.] Describe all of the functions f which are solutions to the differential equation $f'' = -f$ and may be represented by a power series on some interval about $c = 0$.

3. [5pts.] Suppose that f is a differentiable function on an interval A with the property that $|f'(x)| \leq M$ on A. Prove that f is uniformly continuous on A.

4. Compute the derivative functions of the following functions where they exist.

   (a) [2pts.] $f(x) = |x| + |x - 1|$

   (b) [3pts.]

$$g(x) = \begin{cases} \left(\sin^2 x\right) \cdot \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

5. Compute the following limits.

   (a) [2pts.] $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$

   (b) [2pts.] $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

   (c) [2pts.] $\lim_{x\to 0} \frac{1}{e^x-1} - \frac{1}{x}$

6. (a) [4pts.] Suppose that $K \subset \mathbb{R}$ is compact, and $f_n \colon K \to \mathbb{R}$ is a sequence of continuous functions such that $f_n \to f$ pointwise, and such that $f_n(x) \leq f_{n+1}(x)$ for all $x \in K$ and f is continuous. Show that in fact (f_n) converges uniformly.

   [Hint: For $\epsilon > 0$, let K_n be the set of $x \in K$ for which $f(x) - f_n(x) \geq \epsilon$ and consider the sets $K_1, K_2, \ldots$]

   (b) [1pts.] Give an example to show that compactness is necessary in the proposition above. Your example can be either increasing or decreasing; the proposition above works for monotone generally.

7. Consider the sequence of functions $f_n(x) = \frac{x^n}{1+x^n}$.
   (a) [2pts.] What is the pointwise limit of (f_n) on $[0, \infty)$?
   (b) [2pts.] Does (f_n) converge uniformly on $[0, 1]$?
   (c) [2pts.] Does (f_n) converge uniformly on $[2, \infty)$?

8. (a) [3pts.] Compute $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$.

   (b) [2pts.] Estimate $\sin(.2)$ to within $\frac{1}{1000}$.

9. A sequence of functions (f_n) with $f_n\colon A \to \mathbb{R}$ is said to be *compactly convergent* if, for every compact set $K \subset A$, the sequence $f_n\colon K \to \mathbb{R}$ converges uniformly.

   (a) [2pts.] Give an example of a sequence of functions $f_n\colon A \to \mathbb{R}$ with the property that (f_n) is compactly convergent but not uniformly convergent.

   (b) [3pts.] Prove that if (f_n) converges compactly on a domain A and each f_n is continuous at some $c \in A$, then the limit f is continuous at c. Remark: Note that we are not assuming A contains any interval.