**Math 311H**
**Honors Introduction to Real Analysis**

**Final**

**Instructions:** You have three hours to complete the exam. There are nine questions, worth a total of forty-five points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

| Question | Points | Score |
|---|---|---|
| 1 | 4 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 4 | |
| 5 | 5 | |
| 6 | 5 | |
| 7 | 5 | |
| 8 | 6 | |
| 9 | 6 | |
| Total: | 45 | |

1. For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.
   (a) [1pts.] A subset $E \subset \mathbb{R}$ whose limit points are exactly the rational numbers $\mathbb{Q}$.
   (b) [1pts.] A power series that converges uniformly on its interval of convergence.
   (c) [1pts.] A point $x \in \left(0, \frac{\pi}{2}\right)$ such that $x \geq \tan x$.
   (d) [1pts.] A nonempty connected set which contains no nonempty compact subset.

2. [5pts.] Let $(f_n)$ be a sequence of functions on an interval $(a, b)$ such that each $f_n$ is uniformly continuous on $(a, b)$. Suppose that $f_n \to f$ uniformly on $(a, b)$. Prove or disprove: $f$ is also uniformly continuous on $(a, b)$.

3. (a) [3pts.] Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with the property that infinitely many $a_n$ are integers. Prove that the series must have radius of convergence $R \leq 1$.

   (b) [2pts.] Give an example of a power series of the form above with $R = 1$.

4. [4pts.] A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a fixed point if $f(x) = x$. Prove that if $f$ is differentiable on $A$ with $f'(x) \neq 1$ for all $x$, then $f$ has at most one fixed point on $A$.

5. (a) [3pts.] Compute $\sum_{n=2}^{\infty} \frac{n^2}{3^n}$.

   (b) [2pts.] Estimate $\frac{1}{e}$ to within $\frac{1}{100}$.

6. Consider the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$.

   (a) [2pts.] At what points does the series converge? Is the convergence conditional or absolute?

   (b) [3pts.] Prove that the series converges uniformly on every bounded interval. [Hint: Consider derivatives.]

7. Compute the derivative functions of the following functions, where they exist.

   (a) [3pts.] $g(x) = xe^{|x|}$

   (b) [2pts.]

$$f(x) = \begin{cases} x \sin x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

8. Compute the following limits.
   (a) [2pts.] $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$
   (b) [2pts.] $\lim_{x\to 0^+} x^x$
   (c) [2pts.] $\lim_{x\to 0} \frac{1-\cos x}{e^x-1}$

9. A sequence of functions $f_n\colon A \to \mathbb{R}$ is said to be *equicontinuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x - y| < \delta$ implies that $|f_n(x) - f_n(y)| < \epsilon$ for all $x, y \in A$ and all $n$.

   (a) [2pts.] Give an example of a pointwise convergent sequence of functions $f_n\colon A \to \mathbb{R}$ such that each $f_n$ is uniformly continuous on $A$ but $(f_n)$ is not equicontinuous on $A$.

   (b) [2pts.] Let $(f_n)$ with $f_n\colon [0,1] \to \mathbb{R}$ be equicontinuous and uniformly bounded; that is, there exists $M$ with the property that $|f_n(x)| \leq M$ for all $x \in [0,1]$ and all $n$. Prove $(f_n)$ has a subsequence which converges pointwise at every rational number.

   [Hint: By Bolzano-Weierstrass, there is certainly a subsequence of $(f_n(1))$ which converges. How could you modify this to converge at a second rational?]

   (c) [2pts.] Prove that the subsequence of part (b) converges uniformly on all of $[0,1]$.

   [Hint: $[0,1]$ may be covered by finitely many neighborhoods of length $\delta$ for any $\delta$.]

This page is for scratch work. Please label anything you want graded very clearly.