Math 311: Quiz

February 3, 2021

Instructions

You have thirty minutes to take the quiz. There are three questions, each of which is worth five points. You should not use any notes, books, websites, or other aids. After time is called, please upload your solutions, after which you will be asked to record a brief video of yourself explaining one of your solutions for authentication purposes.

Problem 1

Use induction to prove that $7^n - 6n - 1$ is divisible by 36 for all $n \geq 2$.

Solution Let P_n be the statement $36|(7^n - 6n - 1)$. The base case is that the statement P_2 is true because $7^2 - 6(2) - 1 = 49 - 12 - 1 = 36$. For the inductive step, assume P_n is true, so that $7^n - 6n - 1 = 36k$ for some natural number k. Then we have

$$\begin{aligned} 7^{n+1} - 6(n+1) - 1 &= 7(7^n) - 6n - 7 \\ &= 7(7^n - 6n - 1) + 36n \\ &= 7(36k) + 36n \\ &= 36(7k + n) \end{aligned}$$

So P_n implies P_{n+1}, and we are done.

Problem 2

If A and B are sets, their *product* is the set $A \times B = \{(a, b) : a \in A, b \in B\}$. Prove that if

$$\begin{aligned} A &= \{a_1, a_2, a_3, \dots\} \\ B &= \{b_1, b_2, b_3, \dots\} \end{aligned}$$

are both countable sets, then their product $A \times B = \{(a_i, b_j) : i, j \in \mathbb{N}\}$ is as well. (Hint: Your proof should look similar to the proof that $\mathbb{Q}$ is countable from class.)

Solution We may construct a bijection between $A \times B$ and the natural numbers by making a grid of elements (a_i, b_j) and listing the elements along the diagonals, obtaining

$$(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_3, b_1), (a_2, b_2), (a_1, b_3), \dots$$

as the resulting list of elements that defines the bijection.

Problem 3

Let A be a nonempty subset of $\mathbb{R}$ which is bounded above. Let s be a real number with the property that if $n \in \mathbb{N}$, then $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Prove that $s = \sup A$.

Solution First we claim that s is an upper bound for A. For suppose not, then there is some $a \in A$ such that $s < a$. But we may choose $n \in \mathbb{N}$ such that $\frac{1}{n} < a - s$, so that $s + \frac{1}{n} < a$. This is a contradiction, since we assumed that $s + \frac{1}{n}$ was an upper bound for A for all $n \in \mathbb{N}$. So s is an upper bound for A. Now, suppose that there is an upper bound b for A such that $b < s$. Then choose $n \in \mathbb{N}$ such that $\frac{1}{n} < s - b$. Then rearranging we have $b < s - \frac{1}{n}$, so since b is an upper bound for A, for all $a \in A$ we have $a \leq b < s - \frac{1}{n}$. This is a contradiction since we assumed that $s - \frac{1}{n}$ was not an upper bound for A. Ergo there does not exist an upper bound for A which is less than s, implying that for any upper bound b for A, $s \leq b$. So s is the supremum of A.