Math 311: Midterm 2

March 31, 2021

Instructions

You have sixty minutes to take the exam. There are five questions, each of which is worth five points. You should not use any notes, books, websites, or other aids. After time is called, please upload your solutions, after which you will be asked to record a brief video of yourself explaining one of your solutions for authentication purposes.

Problem 1

For each of the following, give an example of the object described or say why this is impossible.

(a) A sequence of closed sets $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ such that $\bigcap_{n=1}^{\infty} F_n$ is empty.

Consider $F_n = [n, \infty)$, so that $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

(b) A perfect set with exactly three limit points.

Impossible; every point of a perfect set is a limit point and perfect sets are always uncountable.

(c) A bounded below set which does not contain its infimum but is not open.

Consider $(0, 1]$.

(d) A set E with $E^\circ = \emptyset$ and $\overline{E} = \mathbb{R}$.

Consider $\mathbb{Q}$.

Problem 2

(a) Let $K, F \subseteq \mathbb{R}$ such that K is compact and F is open. Prove that $K \cap F^c$ is compact.

K is compact, hence closed and bounded. Since F is open, F^c is closed. Since the intersection of two closed sets is closed, $K \cap F$ is closed; since $K \cap F \subseteq K$, we have also that $K \cap F$ is bounded. Ergo $K \cap F$ is compact.

(b) Let $K_1, K_2 \subseteq \mathbb{R}$ such that K_1 and K_2 are both compact. Prove that $K_1 \cup K_2$ is compact. K_1 and K_2 are both closed, so since finite unions of closed sets are closed, $K_1 \cup K_2$ is closed. Moreover, if K_1 is bounded by M_1, so that every $x \in K_1$ has $|x| < M_1$, and K_2 is similarly bounded by M_2, then $K_1 \cup K_2$ is bounded by $\max\{M_1, M_2\}$, so $K_1 \cup K_2$ is also bounded. Ergo $K_1 \cup K_2$ is compact.

Problem 3

(The Squeeze Theorem for Limits of Functions) Let f, g, h be functions with the same domain A and let $f(x) \leq g(x) \leq h(x)$ for all $x \in A$. If $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$ for c some limit point of A, prove that $\lim_{x\to c} g(x) = L$ as well.

Let (x_n) be a sequence of points in A converging to c such that $x_n \neq c$ for any n. Since $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$, we have that $f(x_n) \to L$ and $h(x_n) \to L$. However, since $f(x_n) \leq g(x_n) \leq h(x_n)$, we see that by the Squeeze Theorem for sequences, the sequence $(g(x_n))$ also converges to L. As (x_n) was arbitrary, $\lim_{x\to c} g(x) = L$.

Problem 4

Let $g : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that $g(x) > 0$. Prove that there is some $V_\delta(x)$ such that for $y \in V_\delta(x)$ we have $g(y) > 0$.

Let $g(x) = \alpha$. Let $\epsilon = \frac{\alpha}{2}$. Then there is some δ such that for $|y-x| < \delta$ we have $|g(y)-\alpha| < \frac{\alpha}{2}$, or $-\frac{\alpha}{2} < g(y) - \alpha < \frac{\alpha}{2}$, that is, $\frac{\alpha}{2} < g(y) < \frac{3\alpha}{2}$. In particular $g(y) > \frac{\alpha}{2} > 0$ for all $y \in V_\delta(x)$.

Problem 5

For each of the following pairs of sets, either give an example of a continuous function $f : A \to \mathbb{R}$ whose image is $f(A) = B$ (no need to justify your answer) or explain why no such function exists.

(a) $A = (0, \infty)$; $B = [1, 2]$.

This is possible, let

$$f(x) = \begin{cases} 1 & x < 1 \\ x & 1 \leq x \leq 2 \\ 2 & x > 2 \end{cases}$$

(b) $A = (0, 1) \cup (2, 3)$; $B = (0, 1) \cup (2, 3) \cup (4, 5]$.
Impossible. By the Intermediate Value Theorem, the image of an interval under a continuous function is an interval, implying that each of $f((0, 1))$ and $f((2, 3))$ are contained in one of the three intervals in B. This leaves an interval in B outside the image of f.

(c) A is the Cantor set; $B = [0, 1) \cup (2, 3]$.

Impossible; A is compact and B is not.