Math 311: Sample Midterm 1

March 31, 2021

Instructions

You have sixty minutes to take the exam. There are five questions, each of which is worth five points. You should not use any notes, books, websites, or other aids. After time is called, please upload your solutions, after which you will be asked to record a brief video of yourself explaining one of your solutions for authentication purposes.

Problem 1

For each of the following, give an example of the object described (no need to justify your work) or say why this is impossible.

(a) A perfect set consisting only of rational numbers.

Impossible; perfect sets are always uncountable.

(b) A sequence of open sets $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ such that $\bigcap_{n=1}^{\infty} F_n$ is empty.

Consider $F_n = (0, \frac{1}{n})$, which has $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

(c) A closed set E for which $\overline{(E^\circ)} \neq E$.

Consider $E = [0,1] \cup \{2\}$, so that $E^\circ = (0,1)$ and $\overline{(E^\circ)} = [0,1]$.

(d) A bounded above set A which contains its supremum but is not closed.

Consider, eg, $(1,2]$.

Problem 2 Let A_n be a subset of $\mathbb{R}$ for every n.

(a) Prove that $\bigcup_{n=1}^{\infty} A_n^\circ \subseteq (\bigcup_{n=1}^{\infty} A_n)^\circ$.

Suppose that $x \in \bigcup_{n=1}^{\infty} A_n^\circ$. Then in particular $x \in A_n^\circ$ for some n, implying that there is some $V_\epsilon(x) \subset A_n$. But then $V_\epsilon(x) \subset \bigcup_{n=1}^{\infty} A_n$, so we see that $x \in (\bigcup_{n=1}^{\infty} A_n)^\circ$. The inclusion follows. (You can also do this by noting that $\bigcup_{n=1}^{\infty} A_n^\circ$ is an open set contained in $\bigcup_{n=1}^{\infty} A_n$ and therefore is a subset of $(\bigcup_{n=1}^{\infty} A_n)^\circ$.)

(b) Give an example to show that the inclusion above need not be an equality of sets.

Consider, eg, $A_1 = \mathbb{R}\ \mathbb{Q}$, and letting the remaining sets $A_n = \{r_{n-1}\}$ each contain a single rational number in some enumeration of the rationals $(r_1, r_2, r_3, \dots)$, so that each set has empty interior but the interior of their union is all of $\mathbb{R}$.

Problem 3

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that if $g(r) = 0$ for all points $r \in \mathbb{Q}$, then in fact $g(x) = 0$ for all $x \in \mathbb{R}$.

For any $x \in \mathbb{R}$, pick a sequence of rational numbers $r_n \to x$. Then by continuity of g, we have $g(r_n) \to g(x)$. But $g(r_n) = 0$ for all n, implying that in fact $g(x) = 0$.

Problem 4

Suppose that f, g are two functions with the same domain A such that $f(x) \leq g(x)$ for all $x \in A$, and say that $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} L_2$ both exist for some limit point c of A. Prove that $L_1 \leq L_2$.

Let (x_n) be a sequence of points in A such that $x_n \to c$ but $x_n \neq c$ for any n. Then $f(x_n) \to L_1$ and $g(x_n) \to L_2$ by the sequential criterion for limits. However, $f(x_n) \leq g(x_n)$ for all n, so by the Order Limit Theorem for sequences, we must have that $L_1 \leq L_2$.

Problem 5

For each of the following pairs of sets, either give an example of a continuous function $f : A \to \mathbb{R}$ whose image is $f(A) = B$ (no need to justify your answer) or explain why no such function exists.

(a) $A = [0, 1]$; $B = [1, 2)$

Impossible; A is compact and B is not.

(b) $A = (0, 1]$; $B = [1, \infty)$

Possible; consider $f(x) = \frac{1}{x}$.

(c) $A = (0, 1)$; $B = (0, 1) \cup (3, 4)$

Impossible; A is connected and B is not.