Math 311: Midterm 2

April 12, 2021

Instructions

You have sixty minutes to take the exam. There are five questions, each of which is worth five points. You should not use any notes, books, websites, or other aids. After time is called, please upload your solutions, after which you will be asked to record a brief video of yourself explaining one of your solutions for authentication purposes.

Problem 1

For each of the following, give an example of the object described or say why this is impossible.

(a) A sequence of closed sets $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ such that $\bigcap_{n=1}^{\infty} F_n$ is empty.

(b) A perfect set with exactly three limit points.

(c) A bounded below set which does not contain its infimum but is not open.

(d) A set E with $E^\circ = \emptyset$ and $\overline{E} = \mathbb{R}$.

Problem 2

(a) Let $K, F \subseteq \mathbb{R}$ such that K is compact and F is open. Prove that $K \cap F^c$ is compact.

(b) Let $K_1, K_2 \subseteq \mathbb{R}$ such that K_1 and K_2 are both compact. Prove that $K_1 \cup K_2$ is compact.

Problem 3

(The Squeeze Theorem for Limits of Functions) Let f, g, h be functions with the same domain A and let $f(x) \leq g(x) \leq h(x)$ for all $x \in A$. If $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$ for c some limit point of A, prove that $\lim_{x\to c} g(x) = L$ as well.

Problem 4

Let $g : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that $g(x) > 0$. Prove that there is some $V_\delta(x)$ such that for $y \in V_\delta(x)$ we have $g(y) > 0$.

Problem 5

For each of the following pairs of sets, either give an example of a continuous function $f : A \to \mathbb{R}$ whose image is $f(A) = B$ (no need to justify your answer) or explain why no such function exists.

(a) $A = (0, \infty)$; $B = [1, 2]$.

(b) $A = (0, 1) \cup (2, 3)$; $B = (0, 1) \cup (2, 3) \cup (4, 5]$.

(c) A is the Cantor set; $B = [0, 1) \cup (2, 3]$.