Math 311: Midterm 1

February 18, 2021

Instructions

You have sixty minutes to take the exam. There are five questions, each of which is worth five points. You should not use any notes, books, websites, or other aids. After time is called, please upload your solutions, after which you will be asked to record a brief video of yourself explaining one of your solutions for authentication purposes.

Problem 1

For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.

(a) [1 pt] A Cauchy sequence with no monotone subsequence.

Solution: This is impossible. Cauchy sequences are bounded, and every bounded sequence has a montone subsequence.

(b) [1 pt] A monotone sequence with no Cauchy subsequence.

Consider $(1, 2, 3, 4, \dots)$.

(c) [1 pt] A sequence with exactly three subsequential limits.

Consider $(1, 2, 3, 1, 2, 3, 1, 2, 3, \dots)$.

(d) [1 pt] A series $\sum_{n=1}^{\infty} a_n$ for which $\sum_{n=1}^{\infty} |a_n|$ converges but $\sum_{n=1}^{\infty} a_n$ does not.

Solution: This is impossible; absolutely convergent series converge.

(e) [1 pt] A series whose sum is 3.

Solution: Consider $\sum_{n=1}^{\infty} \frac{3}{2^n}$.

Problem 2

Let (a_n) and (b_n) be sequences of positive real numbers such that $\lim \frac{a_n}{b_n} = L$ is nonzero. Prove that (a_n) is bounded if and only if (b_n) is bounded. [Warning: Be sure not to assume that (a_n) or (b_n) converge.]

Solution: Suppose first that (b_n) is bounded, and pick M such that $b_n < M$ for all n. Then choose N such that $n \geq N$ implies that $|\frac{a_n}{b_n} - L| < 1$, or in particular $\frac{a_n}{b_n} < L + 1$. Then $a_n < (L+1)b_n < (L+1)M$ for all $n \geq N$, which means that if $M' = \max\{a_1, \ldots, a_{N-1}, (L+1)M\}$, then $a_n < M'$ for all n. So (a_n) is bounded. The other direction follows by observing that $\frac{b_n}{a_n} \to \frac{1}{L} \neq 0$ by the Algebraic Limit Theorem and repeating the argument.

Problem 3

Suppose that a_n and b_n are Cauchy sequences. Prove directly that $(a_n b_n)$ is a Cauchy sequence. ["Directly" here means that your proof should not reference the fact that Cauchy sequences converge in $\mathbb{R}$.]

Solution: Cauchy sequences are bounded, so we may choose M_1 such that $|a_n| < M_1$ for all n and $|b_n| < M_2$ for all n. Let $M = \max\{M_1, M_2\}$. Given $\epsilon > 0$, pick N_1 such that $n, m \geq N_1$ implies that $|a_n - a_m| < \frac{\epsilon}{2M}$ and N_2 such that $n, m \geq N_2$ implies that $|b_n - b_m| < \frac{\epsilon}{2M}$. Then $n, m \geq N = \max\{N_1, N_2\}$ implies that

$$\begin{aligned}
|a_n b_n - a_m b_m| &= |a_n b_n - a_n b_m + a_n b_m - a_m b_m| \\
&\leq |a_n b_n - a_n b_m| + |a_n b_m - a_m b_m| \\
&= |a_n||b_n - b_m| + |a_n - a_m||b_m| \\
&< M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2M} \cdot M \\
&= \epsilon.
\end{aligned}$$

Problem 4 Consider the sequence defined recursively by $a_1 = 1$ and $a_n = \frac{5a_n}{3+a_n}$.

(a) [3 pts] Prove that (a_n) is increasing and $1 \leq a_n \leq 2$ for all n.

Solution: We start by showing the bound. Since $a_1 = 1$, the base case $1 \leq a_0 \leq 2$ is clearly true. Now suppose that $1 \leq a_n \leq 2$. We observe that

$$a_{n+1} \geq 1 \Leftrightarrow \frac{5a_n}{3 + a_n} \geq 1 \Leftrightarrow 5a_n \geq 3 + a_n \Leftrightarrow 4a_n \geq 3 \Leftrightarrow a_n \geq \frac{3}{4}$$

where in the second step we use the fact that $3 + a_n$ is positive, so since $a_n \geq 1 > \frac{3}{4}$, we conclude that $a_{n+1} \geq 2$. Likewise

$$a_{n+1} \leq 2 \Leftrightarrow \frac{5a_n}{3 + a_n} \leq 2 \Leftrightarrow 5a_n \leq 6 + 2a_n \Leftrightarrow 3a_n \leq 6 \Leftrightarrow a_n \leq 2$$

so $a_{n+1} \leq 2$ is also true.
Now we check that the sequence is increasing, that is that $a_{n+1} - a_n \geq 0$. Observe that

$$a_{n+1} - a_n = \frac{5a_n}{3 + a_n} - a_n = \frac{5a_n - 3a_n - a_n^2}{3 + a_n} = \frac{(2 - a_n)(a_n)}{3 + a_n}$$

Since $1 \leq a_n \leq 2$, this expression is clearly nonnegative. So $a_{n+1} \geq a_n$ for all n.

(b) [2 pts] Prove that (a_n) converges and compute the limit, justifying your steps carefully.

Solution: From part (a), the sequence (a_n) is bounded monotone, hence convergent. Applying the algebraic limit theorems to the relationship $a_n = \frac{5a_n}{3+a_n}$ we obtain $a = \frac{5a}{3+a}$, or $3a + a^2 = 5a$, which simplifies to $a(a-2) = 0$. By the order limit theorem, the only solution to this which can be the limit of the sequence is 2.

Problem 5

Let $\sum_{n=1}^{\infty} a_n$ be a series with the property that $\lim a_n = 0$.

(a) [1 pt] Give an example to show that $\sum_{n=1}^{\infty} a_n$ need not necessarily converge.

Solution: Consider $\sum_{n=1}^{\infty} \frac{1}{n}$.

(b) [4 pts] Prove that there exists a subsequence (a_{n_k}) of (a_n) with the property that $\sum_{k=1}^{\infty} a_{n_k}$ converges. [Hint: Start by arguing that there is a subsequence (a_{n_k}) with the property that $|a_{n_k}| \leq \frac{1}{k^2}$.]

Solution: Choose a subsequence (a_{n_k}) as follows. Since $a_n \to 0$, there is some a_{n_1} such that $|a_{n_1}| < 1$. Having chosen this n_1, choose n_2 such that $n_1 < n_2$ and $|a_{n_2}| < \frac{1}{4}$, which must exist since otherwise a_n does not converge to zero. Inductively, continue picking a subsequence such that $n_1 < n_2 < n_3 < \dots$ and $|a_{n_k}| < \frac{1}{k^2}$. Then by the Comparison Test, the series $\sum_{k=1}^{\infty} a_{n_k}$ converges.