Math 311: Sample Midterm 1

February 24, 2021

Instructions

You have sixty minutes to take the exam. There are five questions, each of which is worth five points. You should not use any notes, books, websites, or other aids. After time is called, please upload your solutions, after which you will be asked to record a brief video of yourself explaining one of your solutions for authentication purposes.

Problem 1

For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.

(a) [1 pt] A sequence which is not monotone and has no convergent subsequence.

Solution: Consider the sequence $(1, -1, 2, -2, 3, -3, \dots)$.

(b) [1 pt] A Cauchy sequence with a divergent subsequence.

Solution: This is impossible. Cauchy sequences are convergent, and every subsequence of a convergent sequence converges.

(c) [1 pt] An alternating series $\sum_{n=1}^{\infty} a_n$ of rational numbers converging to $\sqrt{2}$, and a partial sum of this series which is within .01 of $\sqrt{2}$.

Solution: Consider the series $2 - .6 + .02 - .006 + \dots$ oscillating around the decimal expansion of the square root of 2. Since the error on the partial sums of an alternating series is always less than the absolute value of the next term of the series, we see that based on the fourth partial sum of the series being 1.414 and having error of less than .001, the sum of this series is between 1.414 and 1.415 and in particular within .01 of $\sqrt{2}$.

(d) [1 pt] A sequence with exactly two subsequential limits.

Solution: Consider $(1, -1, 1, -1, \dots)$.

(e) [1 pt] A bounded sequence with no Cauchy subsequence.

Solution: This is impossible. By Bolzano-Weierstrass, a bounded sequence has a subsequence which converges, hence is Cauchy.

Problem 2

Let (a_n) be a bounded sequence and (b_n) be a sequence such that $b_n \to 0$. Prove that $a_n b_n \to 0$. [Warning: The Algebraic Limit Theorem doesn't apply to this situation.]

Solution: Let M be a bound on a_n, so that $|a_n| < M$ for all M. Then given $\epsilon > 0$, choose N such that $n \geq N$ implies that $|b_n| < \frac{\epsilon}{M}$. Then for $n \geq N$, we have $|a_n b_n - 0| = |a_n||b_n| < M \cdot \frac{\epsilon}{M} = \epsilon$. So $a_n b_n \to 0$.

Problem 3

Suppose that a_n and b_n are Cauchy sequences. Prove directly that $a_n + b_n$ is a Cauchy sequence. ["Directly" means your proof should not reference the fact that Cauchy sequences converge in $\mathbb{R}$.]

Solution: Let $\epsilon > 0$. Choose N_1 such that $n, m \geq N_1$ implies that $|a_n - a_m| < \frac{\epsilon}{2}$ and N_2 such that $n, m \geq N_2$ implies that $|b_n - b_m| < \frac{\epsilon}{2}$. Then for $n, m \geq \max\{N_1, N_2\}$, we have

$$\begin{aligned}
|(a_n + b_n) - (a_m + b_m)| &= |(a_n - a_m) + (b_n - b_m)| \\
&\leq |a_n - a_m| + |b_n - b_m| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Since ϵ was arbitary we are done.

Problem 4 Consider the sequence defined recursively by $a_0 = 1$ and $a_{n+1} = 2(a_n)^{\frac{2}{3}}$. Prove that this sequence converges and find the limit.

Solution: First we claim that $a_n \leq 8$ for all n. Certainly this is true for the base case $n = 0$ since $a_0 = 1$. Suppose for the inductive step that $a_n \leq 8$. Then $a_{n+1} = 2a_n^{\frac{2}{3}} \leq 2(8)^{\frac{2}{3}} = 2(4) = 8$, as desired.
Next we claim that (a_n) is increasing, that is that $a_n \leq a_{n+1}$. Indeed, we do not need an induction for this step: if $a_n \leq 8$ then $a_n^{\frac{1}{3}} \leq 2$, so $a_{n+1} = 2a_n^{\frac{2}{3}} \geq a_n^{\frac{1}{3}} a_n^{\frac{2}{3}} = a_n$.
Since (a_n) is bounded above and increasing, it must converge to some limit $a > 1$. Applying the algebraic limit theorems to the relationship $a_n^3 = 8a_n^2$, we see that $a^3 = 8a^2$, or $a = 8$.

Problem 5

Let $\sum_{n=1}^{\infty} a_n$ be a series with the property that $\lim |a_n|^{\frac{1}{n}}$ exists and is equal to $L < 1$. Prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely. [This result is called the *Root Test.*]

Solution: We may compare to a geometric series, as follows. Choose k such that $L < k < 1$. Then there is some N such that $n \geq N$ implies that $|a_n|^{\frac{1}{n}} < k$, or in other words $|a_n| < k^n$ for $n \geq N$. By the Comparison Test, $\sum_{n+N}^{\infty} a_n$ converges absolutely. However, this implies that $\sum_{n=1}^{\infty} a_n$ converges absolutely. So we are done.