Math 311: Midterm 1

March 1, 2021

Instructions

You have sixty minutes to take the exam. There are five questions, each of which is worth five points. You should not use any notes, books, websites, or other aids. After time is called, please upload your solutions, after which you will be asked to record a brief video of yourself explaining one of your solutions for authentication purposes.

Problem 1

For each of the following things, either give an example of the described object (no need to justify it) or write a sentence saying why this is impossible.

(a) [1 pt] A Cauchy sequence with no monotone subsequence.

(b) [1 pt] A monotone sequence with no Cauchy subsequence.

(c) [1 pt] A sequence with exactly three subsequential limits.

(d) [1 pt] A series $\sum_{n=1}^{\infty} a_n$ for which $\sum_{n=1}^{\infty} |a_n|$ converges but $\sum_{n=1}^{\infty} a_n$ does not.

(e) [1 pt] A series whose sum is 3.

Problem 2

Let (a_n) and (b_n) be sequences of positive real numbers such that $\lim \frac{a_n}{b_n} = L$ is nonzero. Prove that (a_n) is bounded if and only if (b_n) is bounded. [Warning: Be sure not to assume that (a_n) or (b_n) converge.]

Problem 3

Suppose that a_n and b_n are Cauchy sequences. Prove directly that $(a_n b_n)$ is a Cauchy sequence. ["Directly" here means that your proof should not reference the fact that Cauchy sequences converge in $\mathbb{R}$.]

Problem 4 Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \frac{5a_n}{3+a_n}$.

(a) [3 pts] Prove that $1 \leq a_n \leq 2$ for all n and (a_n) is increasing.

(b) [2 pts] Prove that (a_n) converges and compute the limit, justifying your steps carefully.

Problem 5

Let $\sum_{n=1}^{\infty} a_n$ be a series with the property that $\lim a_n = 0$.

(a) [1 pt] Give an example to show that $\sum_{n=1}^{\infty} a_n$ need not necessarily converge.

(b) [4 pts] Prove that there exists a subsequence (a_{n_k}) of (a_n) with the property that $\sum_{k=1}^{\infty} a_{n_k}$ converges. [Hint: Start by arguing that there is a subsequence (a_{n_k}) with the property that $|a_{n_k}| \leq \frac{1}{k^2}$.]