Problem 3.2.2

First we consider \( A = \{ (-1)^n + \frac{2}{n} : n \in \mathbb{N} \} \).

(d) Recall from last week that the limit points of \( A \) are \(-1\) and \(1\). Since \(1\) is already in the set for \(n = 2\), the closure of \( A \) is \( \overline{A} = A \cup \{-1\} \).

Next we consider \( B = \{ x \in \mathbb{Q} : 0 < x < 1 \} \).

(d) By part (a) from last week, the limit points of \( B \) are all of the points in \([0, 1]\) and there are no isolated points. Ergo, the closure of \( B \) is \( \overline{B} = [0, 1] \).

Problem 3.2.11

(a) We claim that for sets \( A, B \in \mathbb{R} \), we have \( \overline{A \cup B} = \overline{A} \cup \overline{B} \). First suppose \( x \in \overline{A \cup B} \). If \( x \in \overline{A} \), either \( x \in A \) or \( x \) is a limit point of \( A \). If \( x \in A \), then clearly \( x \in A \cup B \), hence \( x \in \overline{A} \cup \overline{B} \).

If \( x \) is a limit point of \( A \), there is a sequence of points \((a_n)\) in \( A \) with \( a_n \neq x \) for any \( n \) such that \( \lim a_n = x \). Then \((a_n)\) is also a sequence of points in \( A \cup B \) with the same properties, hence we see that \( x \) is also a limit point of \( A \cup B \), and therefore \( x \in \overline{A \cup B} \). So \( \overline{A} \subseteq \overline{A \cup B} \).

In the other direction, suppose that \( x \in \overline{A \cup B} \). Again, if \( x \in A \cup B \) then one of \( x \in A \) and \( x \in B \) is true, hence \( x \in \overline{A} \cup \overline{B} \). Now suppose that \( x \) is a limit point of \( A \cup B \). Then there is some sequence \( (c_n) \) of points in \( A \cup B \) such that \( c_n \neq x \) for any \( x \) and \( \lim c_n = x \). Now, we see that \( \{ n : c_n \in A \} \cup \{ n : c_n \in B \} = \mathbb{N} \), so at least one of these two sets of indices is infinite. Suppose without loss of generality that \( \{ n : c_n \in A \} \) is infinite. Then there is a subsequence \( (c_{n_k}) \) of \( (c_n) \) consisting of all the entries \( c_n \) of the sequence which lie in \( A \). Since subsequences of a convergent sequence all converge to the limit of the sequence, we see that \( \lim c_{n_k} = x \). Moreover, \( c_{n_k} \neq x \) for all \( n_k \). So \( x \) is a limit point of \( A \) and \( x \in \overline{A} \subseteq \overline{A \cup B} \). We conclude that \( \overline{A \cup B} \subseteq \overline{A} \cup \overline{B} \).

Since we now have inclusions in both directions, we have proved that \( \overline{A \cup B} = \overline{A} \cup \overline{B} \).

(b) This result does not extend to infinite unions. For example, let \( A_n = \frac{1}{n} \). Then \( \overline{A_n} = \{ \frac{1}{n} \} \) since finite sets have no limit points, but \( \bigcup_{n=1}^{\infty} A_n = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \{1\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = (\bigcup_{n=1}^{\infty} \overline{A_n}) \cup \{1\} \).
Problem 3.2.13

Suppose that \( A \subset \mathbb{R} \) is a nonempty set which is both closed and open, and is not all of \( \mathbb{R} \). Then we can find some \( x \in \mathbb{R} \) such that \( x \notin A \). Observe that \( B = A \cap (-\infty, x) = A \cap [-\infty, x] \) is still both closed and open, since finite intersections of open sets are open and finite intersections of closed sets are closed. If \( B \) is nonempty, since \( B \) is bounded above it has a supremum in \( \mathbb{R} \), call it \( y \). Since \( B \) is closed, \( y \in B \). Since \( B \) is open, \( y \notin B \). This is a contradiction. If \( B \) is empty, we have \( A \) bounded below by \( x \), and we may repeat this argument with the infimum of \( B \), again obtaining a contradiction. So the original assumption that it was possible to find \( x \notin A \) for \( A \) nonempty is false. Hence \( A \) is either \( \emptyset \) or \( \mathbb{R} \).

Problem 3.2.14

(a) Recall that \( \overline{E} \) is the union of \( E \) and the set \( L \) of limit points of \( E \). But \( E \) is closed if and only if \( E \) contains all its limit points, or equivalently if \( L \subset E \) and therefore \( \overline{E} = E \cup L = E \). So we are done.

Similarly, \( E^o \) is the set of points \( x \in E \) with the property that there is some \( \epsilon > 0 \) such that \( V_{\epsilon}(x) \subset E \). But \( E \) is open if and only if every \( x \in E \) has this property, or in other words if and only if \( E^o = E \).

(b) Let \( E \subseteq \mathbb{R} \). Since \( \overline{E} \) is closed and contains \( E \), \( (\overline{E})^c \) is an open set contained in \( E^c \). Therefore in particular, \( (\overline{E})^c \subseteq (E^c)^o \). Now by the same token, \( (E^c)^o \) is an open set contained in \( E^c \), so \( ((E^c)^o)^c \) is a closed set containing \( (E^c)^c = E \), hence contains \( \overline{E} \). So \( \overline{E} \subseteq ((E^c)^o)^c \), implying that \( (E^c)^o \subseteq \overline{E}^c \). Ergo we see that \( (\overline{E})^c = (E^c)^o \).

For the other statement, again start with \( E \subset \mathbb{R} \). Let \( F = E^c \). Then by the preceding part, \( \overline{F}^c = (F^c)^o \), so we have that \( \overline{E}^c = E^o \). Taking the complement of both sides we conclude that \( \overline{E}^c = (E^o)^c \).

Section 3.3

Problem 3.3.1

Suppose that \( K \subset \mathbb{R} \) is compact and nonempty. Then \( K \) is bounded, so \( K \) has a supremum and infimum. Moreover \( K \) is closed, and a closed bounded set contains its supremum and infimum, so \( K \) in fact contains its supremum and infimum.

Problem 3.3.2

(a) The set \( \mathbb{N} \) is not compact. The sequence \((1, 2, 3, \ldots)\) has no subsequence converging in \( \mathbb{N} \), or indeed converging at all.

(b) The set \( A = \mathbb{Q} \cap [0, 1] \) is not compact. The sequence \((.3, .31, .314, .3141, \ldots)\) whose limit is \( \frac{\pi}{10} \) has no subsequence converging in \( A \).

(c) The Cantor set is compact, since it is closed and bounded in \( \mathbb{R} \).
(d) The set \( A = \{ a_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N} \} \) is not compact. The sequence \((a_n)\) has no subsequence converging in \( A \).

(e) The set \( A = \{ 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \} \) is compact. It is clearly bounded. As for whether it is closed, notice that a convergent sequence of points \((a_n)\) in \( A \) has a convergent monotone subsequence \((a_{n_k})\). If \((a_{n_k})\) is not eventually constant, then after possibly deleting repeated terms it must be a subsequence of \( \left( \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \right) \), and therefore converge to 1. So 1 is the only limit point of \( A \). And \( 1 \in A \), so in fact \( A \) is closed, hence compact since it is also bounded.

**Problem 3.3.11**

(a) The open cover \( \{ O_n = (n - \frac{1}{2}, n + \frac{1}{2}) : n \in \mathbb{N} \} \) of \( \mathbb{N} \) has no finite subcover, since each \( O_n \) contains only a single point of the infinite set \( \mathbb{N} \).

(b) Consider the sets

\[
O_1 = \mathbb{Q} \cap ((-1, .3) \cup (.4, 2)) \\
O_2 = \mathbb{Q} \cap ((-1, .31) \cup (.32, 2)) \\
O_3 = \mathbb{Q} \cap ((-1, .314) \cup (.315, 2))
\]

and so on, so that \( O_n \) is missing all rationals within an interval of length \( \frac{1}{10^n} \) containing \( \frac{\pi}{10} \) but all rationals in the interval \([0, 1]\) fall within \( O_n \) for sufficiently large \( n \). This open cover of \( A = \mathbb{Q} \cap [0, 1] \) does not have a finite subcover - if it did, then since \( O_1 \subset O_2 \subset O_3 \subset \ldots \), we would have that \( A \subset O_N \) for some \( N \), which is plainly false.

(d) For the set \( A = \{ a_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N} \} \), we may consider the open cover \( \{ O_n = \left( a_n - \frac{1}{2(n+1)^2}, a_n + \frac{1}{2(n+1)^2} \right) : n \in \mathbb{N} \} \). Then each of the open sets \( O_n \) contains a single point of the infinite set \( A \), hence there is no finite subcover.