Homework 8 Solutions

March 12, 2021

Section 3.2

Problem 3.2.2

First we consider $A = \left\{(-1)^n + \frac{2}{n} : n \in \mathbb{N}\right\}$.

(d) Recall from last week that the limit points of A are -1 and 1. Since 1 is already in the set for $n = 2$, the closure of A is $\overline{A} = A \cup \{-1\}$.
Next we consider $B = \{x \in \mathbb{Q} : 0 < x < 1\}$.

(d) By part (a) from last week, the limit points of B are all of the points in $[0, 1]$ and there are no isolated points. Ergo, the closure of B is $\overline{B} = [0, 1]$.

Problem 3.2.11

(a) We claim that for sets $A, B \in \mathbb{R}$, we have $\overline{A} \cup \overline{B} = \overline{A \cup B}$. First suppose $x \in \overline{A} \cup \overline{B}$. If $x \in \overline{A}$, either $x \in A$ or x is a limit point of A. If $x \in A$, then clearly $x \in A \cup B$, hence $x \in \overline{A \cup B}$. If x is a limit point of A, there is a sequence of points (a_n) in A with $a_n \neq x$ for any n such that $\lim a_n = x$. Then (a_n) is also a sequence of points in $A \cup B$ with the same properties, hence we see that x is also a limit point of $A \cup B$, and therefore $x \in \overline{A \cup B}$. So $\overline{A} \subseteq \overline{A \cup B}$. Similarly $\overline{B} \subseteq \overline{A \cup B}$.

In the other direction, suppose that $x \in \overline{A \cup B}$. Again, if $x \in A \cup B$ then one of $x \in A$ and $x \in B$ is true, hence $x \in \overline{A} \cup \overline{B}$. Now suppose that x is a limit point of $A \cup B$. Then there is some sequence (c_n) of points in $A \cup B$ such that $c_n \neq x$ for any x and $\lim c_n = x$. Now, we see that $\{n : c_n \in A\} \cup \{n : c_n \in B\} = \mathbb{N}$, so at least one of these two sets of indices is infinite. Suppose without loss of generality that $\{n : c_n \in A\}$ is infinite. Then there is a subsequence (c_{n_k}) of (c_n) consisting of all the entries c_n of the sequence which lie in A. Since subsequences of a convergent sequence all converge to the limit of the sequence, we see that $\lim c_{n_k} = x$. Moreover, $c_{n_k} \neq x$ for all n_k. So x is a limit point of A and $x \in \overline{A} \subseteq \overline{A} \cup \overline{B}$. We conclude that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Since we now have inclusions in both directions, we have proved that $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

(b) This result does not extend to infinite unions. For example, let $A_n = \frac{1}{n}$. Then $\overline{A}_n = \{\frac{1}{n}\}$ since finite sets have no limit points, but $\overline{\bigcup_{n=1}^{\infty} A_n} = \overline{\left\{\frac{1}{n} : n \in \mathbb{N}\right\}} = \{1\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \left(\bigcup_{n=1}^{\infty} \overline{A}_n\right) \cup \{1\}$.

Problem 3.2.13

Suppose that $A \subset \mathbb{R}$ is a nonempty set which is both closed and open, and is not all of $\mathbb{R}$. Then we can find some $x \in \mathbb{R}$ such that $x \notin A$. Observe that $B = A \cap (-\infty, x) = A \cap [-\infty, x]$ is still both closed and open, since finite intersections of open sets are open and finite intersections of closed sets are closed. If B is nonempty, since B is bounded above it has a supremum in $\mathbb{R}$, call it y. Since B is closed, $y \in B$. Since B is open, $y \notin B$. This is a contradiction. If B is empty, we have A bounded below by x, and we may repeat this argument with the infimum of B, again obtaining a contradiction. So the original assumption that it was possible to find $x \notin A$ for A nonempty is false. Hence A is either $\emptyset$ or $\mathbb{R}$.

Problem 3.2.14

(a) Recall that $\overline{E}$ is the union of E and the set L of limit points of E. But E is closed if and only if E contains all its limit points, or equivalently if $L \subset E$ and therefore $\overline{E} = E \cup L = E$. So we are done.

Similarly, E° is the set of points $x \in E$ with the property that there is some $\epsilon > 0$ such that $V_\epsilon(x) \subseteq E$. But E is open if and only if every $x \in E$ has this property, or in other words if and only if $E^\circ = E$.

(b) Let $E \subseteq \mathbb{R}$. Since $\overline{E}$ is closed and contains E, $(\overline{E})^c$ is an open set contained in E^c. Therefore in particular, $(\overline{E})^c \subseteq (E^c)^\circ$. Now by the same token, $(E^c)^\circ$ is an open set contained in E^c, so $((E^c)^\circ)^c$ is a closed set containing $(E^c)^c = E$, hence contains $\overline{E}$. So $\overline{E} \subseteq ((E^c)^\circ)^c$, implying that $(E^c)^\circ \subseteq \overline{E}^c$. Ergo we see that $(\overline{E})^c = (E^c)^\circ$.

For the other statement, again start with $E \subset \mathbb{R}$. Let $F = E^c$. Then by the preceding part, $\overline{F}^c = (F^c)^\circ$, so we have that $\overline{E^c}^c = E^\circ$. Taking the complement of both sides we conclude that $\overline{E^c} = (E^\circ)^c$.

Section 3.3

Problem 3.3.1

Suppose that $K \subset \mathbb{R}$ is compact and nonempty. Then K is bounded, so K has a supremum and infimum. Moreover K is closed, and a closed bounded set contains its supremum and infimum, so K in fact contains its supremum and infimum.

Problem 3.3.2

(a) The set $\mathbb{N}$ is not compact. The sequence $(1, 2, 3, \dots)$ has no subsequence converging in $\mathbb{N}$, or indeed converging at all.

(b) The set $A = \mathbb{Q} \cap [0, 1]$ is not compact. The sequence $(.3, .31, .314, .3141, \dots)$ whose limit is $\frac{\pi}{10}$ has no subsequence converging in A.

(c) The Cantor set is compact, since it is closed and bounded in $\mathbb{R}$.

(d) The set $A = \left\{a_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$ is not compact. The sequence (a_n) has no subsequence converging in A.

(e) The set $A = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$ is compact. It is clearly bounded. As for whether it is closed, notice that a convergent sequence of points (a_n) in A has a convergent monotone subsequence (a_{n_k}). If (a_{n_k}) is not eventually constant, then after possibly deleting repeated terms it must be a subsequence of $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right)$, and therefore converge to 1. So 1 is the only limit point of A. And $1 \in A$, so in fact A is closed, hence compact since it is also bounded.

Problem 3.3.11

(a) The open cover $\left\{O_n = \left(n - \frac{1}{2}, n + \frac{1}{2}\right) : n \in \mathbb{N}\right\}$ of $\mathbb{N}$ has no finite subcover, since each O_n contains only a single point of the infinite set $\mathbb{N}$.

(b) Consider the sets

$$\begin{aligned} O_1 &= \mathbb{Q} \cap ((-1, .3) \cup (.4, 2)) \\ O_2 &= \mathbb{Q} \cap ((-1, .31) \cup (.32, 2)) \\ O_3 &= \mathbb{Q} \cap ((-1, .314) \cup (.315, 2)) \end{aligned}$$

and so on, so that O_n is missing all rationals within an interval of length $\frac{1}{10^n}$ containing $\frac{\pi}{10}$ but all rationals in the interval $[0, 1]$ fall within O_n for sufficiently large n. This open cover of $A = \mathbb{Q} \cap [0, 1]$ does not have a finite subcover - if it did, then since $O_1 \subset O_2 \subset O_3 \subset \dots$, we would have that $A \subset O_N$ for some N, which is plainly false.

(d) For the set $A = \left\{a_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$, we may consider the open cover $\left\{O_n = \left(a_n - \frac{1}{2(n+1)^2}, a_n + \frac{1}{2(n+1)^2}\right) : n \in \mathbb{N}\right\}$. Then each of the open sets O_n contains a single point of the infinite set A, hence there is no finite subcover.