Homework 6 Solutions

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Section 2.7

Problem 2.7.2

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ converges by observing that $\frac{1}{2^n+n} < \frac{1}{2^n}$ and applying the Comparison Test.

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges by the Comparison Test since $\left|\frac{\sin(n)}{n^2}\right| \leq \frac{1}{n^2}$.

(c) Diverges; notice that the absolute values of the terms are $\frac{n+1}{2n}$ which converges to $\frac{1}{2}$ rather than 0.

(d) Diverges. Let (s_m) be the partial sums of the series. Notice that $s_{3k} > 1+\frac{1}{4}+\frac{1}{7}+\frac{1}{3k-2}$. If t_m are the partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{3n-2}$, then $s_{3k} = t_k$. And $\sum_{n=1}^{\infty} \frac{1}{3n-2}$ clearly diverges, for example by applying limit comparison to $\sum \frac{1}{n}$, so the terms t_k are unbounded above, implying that the partial sums s_{3k} are unbounded above and our original series diverges.

(e) Diverges. Let (s_m) be the partial sums of the series, (t_m) be the sums of the divergent series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ and (r_m) be the sums of the convergent series $\sum_{n+1}^{\infty} \frac{1}{(2n)^2}$. (Both of these assertions can be quickly confirmed by limit comparison to the obvious thing.) Then the sums t_m are unbounded above and the sums r_m are bounded above, say by some M. We have that $s_{2m} = t_m - r_m$. Given any natural number N, choose m such that $t_m > N + M$, so that $s_{2m} = t_m - r_m > N$. This shows the partial sums s_{2m} are not bounded above. We conclude the series diverges.

Problem 2.7.5

For the series $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$, consider the sequence defined by $\sum_{k=1}^{\infty} 2^k b_{2^k} = \sum_{k=1}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=1}^{\infty} (2^{1-p})^k$. This is geometric and converges exactly if $2^{1-p} < 1$, or equivalently when $p > 1$. Therefore, by the Cauchy condensation test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges exactly when $p > 1$.

Problem 2.7.8

(a) True. Suppose $\sum a_n$ converges absolutely, so that $\sum |a_n|$ converges. Then $a_n \to 0$. There is therefore some N such that $n \geq N$ implies that $|a_n| < 1$. Then for $n \geq N$, we have that $|a_n|^2 < |a_n|$, implying by the Comparison Test that $\sum |a_n|^2$ converges.

(b) False. Consider $\sum a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ and $(b_n) = \left(\frac{(-1)^{n+1}}{\sqrt{n}}\right)$. It's true if you assume absolute convergence, though: (b_n) is bounded by some M and $|a_n b_n| \leq M|a_n|$, so $a_n b_n$ converges by comparison if $\sum |a_n|$ converges.

(c) True. For suppose that $\sum n^2 a_n$ converges. Then $n^2 a_n \to 0$, so there is some N such that $n \geq N$ implies that $|n^2 a_n| < 1$, or in particular $|a_n| < \frac{1}{n^2}$. This implies that $\sum a_n$ converges absolutely. So if $\sum a_n$ converges conditionally, it must be the case that $\sum n^2 a_n$ diverges.

Other Problems

Problem 6

(a) Let $b_n = \frac{1}{n!}$. Then $|\frac{b_{n+1}}{b_n}| = |\frac{1}{n+1}|$, so $\lim |\frac{b_n+1}{b_n}| = 0$. Ergo $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{n!}$ converges absolutely by the Ratio Test.

(b) We compute that

$$\begin{aligned}
a_n &= (1 + \frac{1}{n})^n \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{1}{n}\right)^k \\
&= \sum_{k=0}^{n} \frac{n!}{(n-k)!} \left(\frac{1}{n}\right)^k \frac{1}{k!} \\
&= \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!}
\end{aligned}$$

Observe that for $n \geq 1$, $\frac{n(n-1)\cdots(n-k+1)}{n^k} \leq 1$, so $a_n \leq \frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!} = s_n$. Note that since (s_n) is increasing, this in particular implies $a_n \leq s$ for all $n \geq 1$.

(c) Notice that

$$\begin{aligned}
a_n &= \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!} \\
&= \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!} + \sum_{k=m+1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!} \\
&\geq \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!}
\end{aligned}$$

Fix m and call the righthand side t_n^m. Then as $n \to \infty$, we see that $t_n^m \to 1+1+\frac{1}{2!}+\cdots \frac{1}{m!} = s_m$. In particular, given any ϵ, we observe that there exists some N_m such that $n \geq N_m$ implies that $t_n^m > s_m - \frac{\epsilon}{2}$. Ergo $n \geq N_m$ implies in particular that $a_n > s_m - \frac{\epsilon}{2}$.

(d) Now we complete the proof. Let $\epsilon > 0$. Choose any integer m such that $s - \frac{\epsilon}{2} < s_m \leq s$, which certainly exists since the sequence (s_n) converges to s. Then choose N_m as in part (c) so that $n \geq N_m$ implies $a_n > s_m - \frac{\epsilon}{2}$. Then in total we have

$$s \geq a_n > s_m - \frac{\epsilon}{2} > s - \frac{\epsilon}{2} - \frac{\epsilon}{2} = s - \epsilon.$$

In particular $n \geq N_m$ implies $|a_n - s| < \epsilon$. So $\lim a_n = s$.
This outline is based on the proof given in Rudin's book *Principles of Mathematical Analysis*.