MATH 311: Homework 6

Due: March 3, 2021

1. Upcoming office hours are Monday March 1 9-10 and Wednesday March 3 9-10.

2. Reminder that Midterm 1 is Monday March 1 in class, and covers through the end of Chapter 2. (Explicitly that means through the end of Section 2.7.) The exam will be five questions similar in style to the posted sample midterm. You will have an hour to do it. The protocols for taking it will be the same as the quiz.

3. Read Sections 3.1-2 in Abbott

4. Do exercises 2.7.2, 2.7.5, 2.7.8 in Abbott.

5. *The number e.* You have probably seen in calculus that Euler's number e may be defined as the limit of the sequence $a_n = (1 + \frac{1}{n})^n$. This is sometimes described as the interaction between the "irresistible force" – to wit, an exponent approaching infinity – and the "immovable object" – to wit, a base approaching 1. Another possible definition of e is

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

We will show these expressions are both convergent, and in fact coincide. Let s_n be the partial sums of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$.

- (a) Show that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. Call the limit s.
- (b) The binomial theorem states that, for $n \geq 1$, $(1+x)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k$. With this in mind, show that for $n \geq 1$,

$$a_n = \frac{1}{0!} + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!}$$

Conclude that $a_n \leq s_n$ for all $n \geq 1$. Note that since (s_n) is increasing, this in particular implies $a_n \leq s$ for all $n \geq 1$.

- (c) For $n \geq m$, show that

$$a_n \geq \frac{1}{0!} + \sum_{k=1}^{m} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{1}{k!}.$$

Fix m and call the righthand side t_n^m. Then as $n \to \infty$, we see that $t_n^m \to 1 + 1 + \frac{1}{2!} + \cdots \frac{1}{m!} = s_m$. In particular, given any ϵ, we observe that there exists some N_m such that $n \geq N_m$ implies that $t_n^m > s_m - \frac{\epsilon}{2}$. Ergo $n \geq N_m$ implies in particular that $a_n > s_m - \frac{\epsilon}{2}$.

- (d) Now we complete the proof. Let $\epsilon > 0$. Choose any integer m such that $s - \frac{\epsilon}{2} < s_m \leq s$, which certainly exists since the sequence (s_n) converges to s. Then choose N_m as in part (c) so that $n \geq N_m$ implies $a_n > s_m - \frac{\epsilon}{2}$. Then in total we have

$$s \geq a_n > s_m - \frac{\epsilon}{2} > s - \frac{\epsilon}{2} - \frac{\epsilon}{2} = s - \epsilon.$$

In particular $n \geq N_m$ implies $|a_n - s| < \epsilon$. So $\lim a_n = s$.