Homework 5 Solutions

February 24, 2021

Section 2.5

Problem 2.5.2

(a) True. For example, the subsequence obtained by deleting the first element of the sequence is a proper subsequence of the sequence, and if that subsequence converges, the sequence certainly does.

(b) True. Every subsequence of a convergent sequence converges, so a sequence with a divergent subsequence is necessarily divergent.

(c) True. Any bounded sequence has at least one subsequence converging to some subsequential limit; see the argument of Problem 2.5.5 below for the construction of a second subsequential limit in the case that the sequence diverges.

(d) True. Let (x_n) be increasing and contain a convergent subsequence (x_{n_k}). Then the subsequence is in particular bounded above by some M; that is, $x_{n_k} < M$ for all n_k. For any $n \in \mathbb{N}$ we may choose n_k such that $n < n_k$, and then $a_n \leq a_{n_k} < M$. So (x_n) is bounded above, hence convergent since it is increasing. The case of a decreasing sequence is similar.

Problem 2.5.5

Let (a_n) be a bounded sequence with the property that every convergent subsequence converges to the same limit a. Suppose, for the sake of inducing a contradiction, that (a_n) does not converge to a. Then there is some $\epsilon > 0$ such that for any N a natural number, we can find $n > N$ such that $|a_n - a| \geq \epsilon$. In particular, we may pick a subsequence (a_{n_k}) of (a_n) consisting of terms of distance at least ϵ from a. Now, (a_{n_k}) is a bounded sequence, hence by Bolzano-Weierstrass it has a convergent subsequence (a_{n_j}) with limit some real number b. However, (a_{n_j}) consists solely of points of distance at least ϵ from a, so $|b - a| \geq \epsilon$ and in particular $b \neq a$. So (a_{n_j}) is a subsequence of (a_n) not converging to a. This is a contradiction. So $\lim a_n = a$.

Problem 2.5.6

We wish to compute the limit of $(b^{\frac{1}{n}})$ for all $b \geq 0$. If $b = 0$ this limit obviously exists and is 0; similarly if $b = 1$ the limit obviously exists and is 1.

Now consider the case $0 < b < 1$. We see first that $c_n = (b^{\frac{1}{n}})$ is positive and bounded above by 1. Moreover we claim the sequence is increasing. In particular, $c_n^n = b = c_{n+1}^{n+1}$. Since $c_{n+1} < 1$, this implies that $c_{n+1}^n > b = c_n^n$, which in turn implies that $c_{n+1} > c_n > b$. So the

sequence is bounded monotone, hence convergent. Let the limit be ℓ. Observe that $b \leq \ell \leq 1$ by the order limit theorem. We consider the subsequence $(b^{\frac{1}{2n}})$. By the results of last week's homework, this subsequence converges to $\sqrt{\ell}$. But every subsequence of a convergent sequence converges to the limit of the sequence, so in fact we must have $\ell = \sqrt{\ell}$. This implies $\ell = 1$. So $\lim b^{\frac{1}{n}} = 1$.

The case that $b > 1$ follows by writing $b = \frac{1}{c}$ so that $b = \frac{1}{c^{\frac{1}{n}}}$ and applying the algebraic limit theorem to the quotient. In particular we see that $\lim b^{\frac{1}{n}} = 1$ again.

Section 2.6

Problem 2.6.2

(a) The sequence $(\frac{(-1)^n}{n})$ is convergent, hence Cauchy, but not monotone.

(b) Cauchy sequences are bounded, so every subsequence of a Cauchy sequence is bounded. Hence this is impossible.

(c) Impossible; we claim that a monotone sequence with a Cauchy subsequence must be bounded, hence convergent. To justify this, suppose (a_n) is increasing with a Cauchy subsequence (a_{n_k}). Since Cauchy sequences are bounded there is some M such that $a_{n_k} < M$ for all M. Then for any n, there is some $n_k > n$, so $a_n \leq a_{n_k} \leq M$, so (a_n) is also bounded by M. The case for descreasing sequences is similar.

(d) Consider the sequence $(1, 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \dots)$, which is unbounded but contains the Cauchy sequence $\left(\frac{1}{n}\right)$ as a subsequence.

Problem 2.6.4

Let (a_n) and (b_n) be Cauchy sequences.

(a) Yes, (c_n) where $c_n = |a_n - b_n|$ is Cauchy. Given $\epsilon > 0$, there is some N_1 such that $n, m \geq N_1$ implies $|a_n - a_m| < \frac{\epsilon}{2}$ and some N_2 such that $n, m \geq N_2$ implies that $|b_n - b_m| < \frac{\epsilon}{2}$. Then for $n, m \geq N = \max\{N_1, N_2\}$, we have

$$\begin{aligned} |a_n - b_n| &\leq |a_n - a_m| + |a_m - b_m| + |b_m - b_n| \\ &< |a_m - b_m| + \epsilon. \end{aligned}$$

and similarly $|a_m - b_m| < |a_n - b_n| + \epsilon$, so in total $|a_m - b_m| - \epsilon < |a_n - b_n| < |a_m - b_m| + \epsilon$. In particular $n, m \geq N$ implies $||a_m - b_m| - |a_n - b_n|| < \epsilon$.

(b) The sequence $((-1)^n a_n)$ need not be Cauchy. For example, $a_n = 1$ is a Cauchy sequence but $((-1)^n a_n)) = ((-1)^n)$ is not.

(c) The sequence $([[a_n]])$ need not be Cauchy; consider the Cauchy sequence $a_n = \left(\frac{(-1)^n}{n}\right)$, which has $([[a_n]]) = (-1, 0, -1, 0, \dots)$.

Other Problems

Problem 3

(a) $\{5, -1\}$.

(b) $\{0, \pm\frac{\sqrt{3}}{2}\}$

(c) $\{0\}$

(d) All of $\mathbb{R}$.

Problem 5

Suppose that (a_n) is a sequence such that $|a_n - a_{n+1}| < \frac{1}{2^n}$. Given $\epsilon > 0$, choose N such that $\frac{1}{2^{N-1}} < \epsilon$. Then if we have $n > m \geq N$, we compute that

$$\begin{aligned}
|a_m - a_n| &= |(a_m - a_{m+1}) + (a_{m+1} - a_{m+2}) + \cdots + (a_{n-1} - a_n)| \\
&\leq |(a_m - a_{m+1})| + |(a_{m+1} - a_{m+2})| + \cdots + |(a_{n-1} - a_n)| \\
&< \frac{1}{2^m} + \frac{1}{2^{m-1}} + \cdots + \frac{1}{2^{n-1}} \\
&< \sum_{k=0}^{\infty} \frac{1}{2^m} \cdot \frac{1}{2^k} \\
&= \frac{1}{2^{m-1}} \\
&\leq \frac{1}{2^{N-1}} \\
&< \epsilon
\end{aligned}$$

Since ϵ was arbitary we conclude the sequence is Cauchy.

The claim does not stay true if $\frac{1}{2^n}$ is replaced by $\frac{1}{n}$. For example, the partial sums (s_n) of the series $\sum_{k=1}^{\infty} \frac{1}{k}$ have the property that $|s_n - s_{n+1}| = \frac{1}{n+1} < \frac{1}{n}$, but this sequence of partial sums does not converge and hence is not Cauchy.