

# Homework 4 Solutions

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## Section 2.3

### 2.3.3

We have  $x_n \leq y_n \leq z_n$  with  $\lim x_n = \lim z_n = \ell$ . Note that we can't use the order limit theorem directly because we don't yet know that  $y_n$  converges. However, let  $\epsilon > 0$ . There exists  $N_1$  such that  $n \geq N_1$  implies that  $|x_n - \ell| < \epsilon$  and in particular that  $\ell - \epsilon < x_n$ . Furthermore there exists  $N_2$  such that  $n \geq N_2$  implies that  $|z_n - \ell| < \epsilon$ , and in particular  $z_n < \ell + \epsilon$ . We see that for  $n \geq N = \max\{N_1, N_2\}$  we have

$$\ell - \epsilon < x_n \leq y_n \leq z_n \leq \ell + \epsilon$$

and in particular  $|x_n - \ell| < \epsilon$ . So  $x_n \rightarrow \ell$ .

### 2.3.5

Given sequences  $(x_n)$  and  $(y_n)$ , we take  $(z_n)$  to be the shuffled sequence  $(x_1, y_1, x_2, y_2, \dots)$  with  $z_{2n-1} = x_n$  and  $z_{2n} = y_n$ . First assume that  $\lim x_n = a = \lim y_n$ . Then for any  $\epsilon > 0$  there exists  $N_1$  such that  $n \geq N_1$  implies that  $|x_n - a| < \epsilon$  and  $N_2$  such that  $n \geq N_2$  implies that  $|y_n - a| < \epsilon$ . Now let  $n \geq N = \max\{2N_1, 2N_2\}$ . If  $n$  is odd,  $z_n = x_{\frac{n+1}{2}}$ , so since  $\frac{n+1}{2} \geq \frac{2N_1+2}{2} > N_1$ , we have that  $|z_n - a| = |x_n - a| < \epsilon$ . Similarly if  $n$  is even  $z_n = y_{\frac{n}{2}}$ , so since  $\frac{n}{2} \geq \frac{2N_2}{2} = N_2$ , we have that  $|z_n - a| = |y_{\frac{n}{2}} - a| < \epsilon$ . So in either case  $n \geq N$  implies that  $|z_n - a| < \epsilon$ , and we conclude that  $z_n \rightarrow a$ .

Now assume that  $\lim z_n = a$  for some  $a$ . Then for any  $\epsilon > 0$  there exists  $N$  such that  $n \geq N$  implies that  $|z_n - a| < \epsilon$ . Now for  $n \geq N$ , observe that  $x_n = z_{2n-1}$ . If  $n \geq N$ , we also have that  $2n-1 \geq N$ , so for  $n \geq N$  we have  $|x_n - a| = |z_{2n-1} - a| < \epsilon$ . Hence  $x_n \rightarrow a$ . Likewise if  $n \geq N$  then  $|y_n - a| = |z_{2n} - a| < \epsilon$ , so  $y_n \rightarrow a$ .

### 2.3.10

(a) False! Let  $a_n = (-1)^n$ ,  $b_n = (-1)^n$ . Both  $(a_n)$  and  $(b_n)$  diverge, but their difference  $a_n - b_n = 0$  is a constant sequence converging to 0.

(b) True. Suppose that  $b_n \rightarrow b$ . Let  $\epsilon > 0$ , then there exists  $N$  such that  $n \geq N$  implies that  $|b_n - b| < \epsilon$ . Now by the Triangle Inequality we have

$$|b_n| = |(b_n - b) + b| \leq |b_n - b| + |b|$$

so in particular  $|b_n| - |b| \leq |b_n - b|$ . By the same logic  $|b| - |b_n| \leq |b_n - b|$ . Therefore we see that  $||b_n| - |b|| \leq |b_n - b|$ . In particular, if  $n \geq N$ , we have  $||b_n| - |b|| \leq |b_n - b| < \epsilon$ . Ergo  $|b_n| \rightarrow |b|$ .

(c) True. Let  $\lim a_n = a$  and  $\lim b_n - a_n \rightarrow 0$ . Then by the algebraic limit theorem,  $\lim b_n = \lim(b_n - a_n + a_n) = \lim(b_n - a_n) + \lim a_n = 0 + a = a$ .

(d) True. Let  $a_n \rightarrow 0$ . Then for any  $\epsilon > 0$ , there exists  $N$  such that  $n \geq N$  implies that  $|a_n| < \epsilon$ . So for  $n \geq N$ , we have  $|b_n - b| < |a_n| < \epsilon$ . We conclude that  $b_n \rightarrow b$ .

## 1 Other Problems

### Problem 4

(a) Let  $(a_n)$  be a sequence with  $a_n \neq 0$  such that  $\left| \frac{a_{n+1}}{a_n} \right| = L < 1$ . Choose a real number  $k$  with  $L < k < 1$ . Then if  $\epsilon = k - L$ , there is some  $N$  such that  $n \geq N$  implies that  $\left| \left| \frac{a_{n+1}}{a_n} \right| - L \right| < \epsilon$ , implying that  $\left| \frac{a_{n+1}}{a_n} \right| < L + \epsilon = k$ . In particular for  $n \geq N$ , we have  $\frac{|a_{n+1}|}{|a_n|} < k$ , or in other words  $|a_{n+1}| < k|a_n|$ . Applying this relationship inductively we see that  $0 \leq |a_{n+N}| < k^n |a_N|$ . Since  $k^n \rightarrow 0$ , we have that  $k^n |a_N| \rightarrow 0$ . Hence by the squeeze theorem  $|a_{n+N}| \rightarrow 0$ . Reindexing this becomes  $|a_n| \rightarrow 0$ . But in general if  $|b| \rightarrow 0$  then  $b \rightarrow 0$  (exercise!) so we see that  $a_n \rightarrow 0$ .

(b) Let  $b_n = \frac{a^n}{n^p}$  for  $|a| < 1$  and  $p > 0$ . Then we have

$$\left| \frac{b_{n+1}}{b_n} \right| = \frac{|a| \cdot n^p}{(n+1)^p}$$

and the limit of this term as  $n \rightarrow \infty$  is  $|a| < 1$ . So  $\lim b_n = 0$  by part (a).

(c) Let  $c_n = \frac{a^n}{n!}$ . Then we have

$$\left| \frac{c_{n+1}}{c_n} \right| = \frac{|a|}{n}$$

and the limit of this term as  $n \rightarrow \infty$  is  $0 < 1$ . So  $\lim c_n = 0$  by part (a).

## 2 Section 2.4

### 2.4.1

(a) Let  $x_1 = 3$  and  $x_{n+1} = \frac{1}{4-x_n}$ . First we claim that  $0 \leq x_n \leq 3$  for all  $n$ . This is clearly true for the base case  $x_1 = 3$ . For the inductive step, suppose we know that  $0 \leq x_n \leq 3$ . Then since  $x_n \leq 3$ , we have that  $4 - x_n$  is positive, so  $x_{n+1} = \frac{1}{4-x_n} > 0$ . Moreover, also since  $x_n \leq 3$ , we have that  $4 - x_n \geq 1 > \frac{1}{3}$ , so  $x_{n+1} = \frac{1}{4-x_n} < 3$ . So  $0 \leq x_{n+1} \leq 3$ . In particular, the sequence  $(x_n)$  is bounded below by 0.

Now we claim that  $(x_n)$  is decreasing; that is, we claim that  $x_n \geq x_{n+1}$  for all  $n$ . For the base case, we have  $x_1 = 3 \geq 1 = x_2$ . Now suppose that  $x_n \geq x_{n+1}$ . Then  $4 - x_n \leq 4 - x_{n+1}$ . Since by the first part both numbers are positive, it follows that

$$x_{n+1} = \frac{1}{4-x_n} \geq \frac{1}{4-x_{n+1}} = x_{n+2}$$

and thus by induction the sequence is decreasing as desired. As  $(x_n)$  is bounded below and decreasing,  $\lim x_n = x$  exists.

(b) The sequences  $(x_n)$  and  $(x_{n+1})$  are the same except for being shifted by an index. If one converges the other converges as well to the same limit.

(c) Taking the limit of both sides using the algebraic limit theorem, we obtain  $x = \frac{1}{4-x}$ , or  $x(4-x) = 1$ . We rearrange to  $0 = x^2 - 4x + 1$ . From the quadratic formula, the solutions to this equation are  $2 \pm \sqrt{3}$ . Since we are plainly looking for a number less than  $x_2 = 1$ , we conclude that  $\lim x_n = 2 - \sqrt{3}$ .

## 2.4.2

(a) This argument doesn't work because the algebraic limit theorems only hold for sequences which are known to converge. In particular,  $(y_n)$  definitely doesn't converge; we have  $y_n = 1$  for  $n$  odd and  $y_n = 2$  for  $n$  even. So the step of taking the limit on both sides of the relationship is invalid.

(b) Yes. Let  $y_1 = 1$  and  $y_{n+1} = 3 - \frac{1}{y_n}$ . First we check the sequence is bounded above and monotone increasing.

First, we claim that  $1 \leq y_n \leq 3$  for all  $n$ . This is clearly true in the base case  $y_1 = 1$ . Now for the inductive step suppose that  $1 \leq y_n \leq 3$ , and let us try to show the same is true of  $y_{n+1}$ . Since  $y_n \geq 1$ , we have  $0 < \frac{1}{y_n} \leq 1$ , so  $2 < 3 - \frac{1}{y_n} \leq 3$ . Ergo we have  $1 \leq y_{n+1} \leq 3$ . So by induction the claim is true for all  $n$ . In particular,  $y_n$  is bounded above by 3.

Now we claim that  $y_n$  is increasing; that is, that  $y_n \leq y_{n+1}$  for all  $n$ . For the base case,  $y_1 = 1 \leq 2 = y_2$ . For the inductive step, assume we know that  $y_n \leq y_{n+1}$ . Then since by the argument above both terms are positive, we have  $\frac{1}{y_n} \geq \frac{1}{y_{n+1}}$  and consequently  $y_{n+1} = 3 - \frac{1}{y_n} \leq 3 - \frac{1}{y_{n+1}} = y_{n+2}$ . Therefore the sequence is increasing.

Since  $(y_n)$  is bounded above and increasing,  $(y_n)$  converges to some  $y$ . Therefore we may apply the algebraic limit theorems to both sides of the equation to conclude that  $y = 3 - \frac{1}{y}$ , which rearranges to  $y^2 - 3y + 1 = 0$ . The solutions to this equation are  $\frac{3 \pm \sqrt{5}}{2}$ ; since we are clearly looking for a number greater than 1, we must have  $y = \frac{3 + \sqrt{5}}{2}$ .

## 2.4.8

(a) From class, the partial sums of  $\sum_{n=1}^{\infty} \frac{1}{2^n}$  are of the form

$$s_m = \frac{1 - (\frac{1}{2})^{m+1}}{1 - \frac{1}{2}} - 1$$

or equivalently

$$s_m = \frac{1}{2} \cdot \frac{1 - (\frac{1}{2})^{m+1}}{1 - \frac{1}{2}}.$$

The limit as  $m \rightarrow \infty$  is  $\frac{1}{2} \cdot \left( \frac{1}{1 - \frac{1}{2}} \right) = 1$ . So  $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ .

(b) For the series  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ , observe that

$$\frac{1}{n(n-1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore the partial sums of the series are of the form

$$\begin{aligned}s_m &= \frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{m(m+1)} \\&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m} - \frac{1}{m+1}\right) \\&= 1 - \frac{1}{m+1}\end{aligned}$$

We see that  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ .

(c) For the series  $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$ , observe that

$$\log\left(\frac{n+1}{n}\right) = \log(n+1) - \log(n).$$

Therefore the partial sums of the series are of the form

$$\begin{aligned}s_m &= \log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \cdots + \log\left(\frac{m+1}{m}\right) \\&= (\log(2) - \log(1)) + (\log(3) - \log(2)) + \cdots + (\log(m+1) - \log(m)) \\&= \log(m+1) - \log(1) \\&= \log(m+1) - 0 \\&= \log(m+1)\end{aligned}$$

Since  $\log(m)$  grows unboundedly as  $m$  grows, the series diverges (more precisely, “diverges to infinity.”)